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# Ternary Quadratic Types

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# TERNARY QUADRATIC TYPES

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The first part of this paper deals with the determination of the complete system of concomitants of five or fewer ternary quadratic forms. In the second part of the paper it is shown that this system is irreducible, and that from it may be deduced the irreducible system of ternary quadratic types, thus giving a classification of the irreducible concomitants of any number of ternary quadratics.

## Introduction

The complete systems of concomitants for two and three ternary quadratic forms were first obtained, respectively, by Gordan (1882) and Ciamberlini (1886). Much later (van der Waerden 1923) it was shown that Gordan's system for two quadratics was irreducible, while Ciamberlini's system for three quadratics contained six reducible forms. The earliest writer to consider systems of more than three quadratics in any detail seems to have been Turnbull (1910). He exhibited a complete system for four quadratics in explicit form, and enumerated (without attempting a detailed reduction) the possible cases which arise for consideration in connexion with five quadratics. This latter system is of special interest, since, by Peano's theorem (Grace & Young 1903, p. 359), the only irreducible ternary quadratic types are those which occur in the irreducible systems of five or fewer quadratics, together with the linear alternating invariant of six quadratics, which is the determinant of the coefficients in these forms. Recently (Todd 1948) Turnbull's system for four quadratics has been re-examined, the reducibility of a number of his forms demonstrated, and the irreducibility of the remaining system established. Although this system contains some six hundred forms, its structure is relatively simple, and none of the irreducible concomitants has degree exceeding eight.

The first part of the present paper is devoted to establishing a complete system for five quadratics which, in the second part of the paper, is shown to be irreducible. The reductions are based on the classification of possible forms given by Turnbull, and in order to make the account self-contained there is a certain amount of overlapping with Turnbull's original paper. This overlapping is essential if the present paper is to be intelligible at all, since many of Turnbull's reductions are not pressed far enough, and an account which begins every discussion at the point where Turnbull breaks it off would, in the present case, be too disjointed to follow.

The reductions are effected by using the familiar identities of symbolic algebra. The length of the discussion is due, in part, to the large number of forms which arise for consideration, and also to the fact that, for a comparatively few forms, the actual reductions themselves are of considerable length. It is necessary, particularly in the more complicated cases, to give the work in detail, since the processes involved are by no means obvious.

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## Notation and identities

1. If we have any set of ternary quadratics, which we may denote symbolically by

$$a_x^2 \equiv a_x'^2 \equiv a_x''^2 \equiv \ldots, \quad b_x^2 \equiv b_x'^2 \equiv b_x''^2 \equiv \ldots, \quad \ldots,$$

and if the corresponding contravariants are denoted by

$$u_\alpha^2 \equiv u_{\alpha'}^2 \equiv u_{\alpha''}^2 \equiv \ldots, \quad u_\beta^2 \equiv u_{\beta'}^2 \equiv u_{\beta''}^2 \equiv \ldots, \quad \ldots,$$

then any concomitant of the forms is expressible as the sum of products of symbolic factors of one or other of the forms $a_x$, $a_\beta$, $u_\alpha$, $(abc)$, $(abu)$, $(\alpha\beta\gamma)$, $(\alpha\beta x)$. Just as in the case of two quadratics (Grace & Young 1903, p. 280), these products have the following properties:

(i) If two equivalent symbols $a, a'$ occur in a bracket the product can be transformed into one in which $a, a'$ occur in two brackets, and the pair of symbols $a, a'$ can then be replaced by a single symbol $\alpha$. If two equivalent symbols of type $\alpha$ occur in a bracket the form is reducible.

(ii) A form which contains a factor $a_\alpha$ can be transformed so as to contain $a_\alpha^2$, and hence is reducible.

(iii) Products which contain a factor of type $(abc)$ or $(abu)$ and another factor of type $(\alpha\beta\gamma)$ or $(\alpha\beta x)$ may be neglected, in view of identities such as $(abc)(\lambda\mu\nu) = \dot{\Sigma} a_\lambda b_\mu c_\nu$, the dot indicating a determinantal permutation of $abc$.

(iv) If a product contains two pairs of equivalent symbols, e.g. $a, a'$, and a second product is obtained by permuting these symbols in any manner, then the difference between the products can be expressed in a form in which $a, a'$ are replaced by their convolution $\alpha = (aa')$, and thus in terms of fewer symbols than the original product.

This last property justifies the use of Turnbull's contracted symbolic notation. This consists in replacing each symbol $a, a', a'', \ldots$ by 0, each symbol $b, b', b'', \ldots$ by 1 and so on, and replacing each symbol $\alpha, \alpha', \alpha'', \ldots$ by 0 or **0** according as it appears as a suffix or in a bracket, and similarly with symbols $\beta, \gamma, \ldots$. The contracted notation may not define a unique form, since the sets of equivalent symbols may be permuted in various ways, but all the forms represented by such a product differ from each other by forms expressible in terms of fewer symbols; and hence, since the forms are to be considered in ascending order of complexity, the various forms represented are all equivalent. We shall also follow Turnbull in abbreviating a product such as $0_2 1_2 1_3 2_3 2_x$ to $(0_2 1_3 2_x)$, the bracket being omitted whenever no ambiguity can arise.

2. The five fundamental quadratics will be denoted by $0_x^2, 1_x^2, \ldots, 4_x^2$. Letters such as $i, j, k, l, m, p, q, r, s, t$ will denote unspecified symbols from the set $0, 1, \ldots, 4$, and will generally be used with the following conventions: $i, j, k, l, m$ when used as symbols all cogredient with 0 or all cogredient with **0** will be supposed different from each other, and $p, q, r, s, t$ will generally be supposed different from each other, but not necessarily different from $i, j, k, l, m$. When this convention is suspended it will be stated explicitly. Greek letters $\xi, \eta$ will usually denote arbitrary symbols cogredient with 0 or **0** (as the context will always indicate); for instance, $0_\xi$ may mean $0_1$ or $0_x$ and $\xi_1$ may be $0_1$ or $u_1$.

The fundamental identity

$$(abc)\, d_x = (dbc)\, a_x + (adc)\, b_x + (abd)\, c_x$$

will be constantly used, and denoted by $[g, d_x]$. If it is applied to a symbolic product containing several bracket factors, one of which is $(abc)$, these factors will be supposed to be numbered from left to right, and the particular bracket $(abc)$ affected indicated by a numerical suffix to the symbol $g$. If after transforming such a product certain of the symbolic products on the right break up in an obvious way into products of actual forms, those products so reducing will usually be omitted and the sign of equality replaced by the sign $\equiv$ of equivalence. The symbol $f_1 \equiv f_2$ means that $f_1 - f_2$ is reducible.

The other identities constantly employed are:

$$(ij\xi)(\mathbf{pq}\eta) = \dot{\Sigma} i_p j_q \xi_\eta, \qquad (\mathbf{pp}'\xi)\,\eta_p \zeta_{p'} \equiv 0,$$
$$(ii'\xi)\, i_\eta i'_\zeta = \tfrac{1}{2}\xi_i(\mathbf{i}\eta\zeta), \qquad \xi_p \xi_{p'} \eta_p \zeta_{p'} \equiv 0,$$
$$i_\xi i'_\xi i_\eta i'_\zeta \equiv -\tfrac{1}{2}(\mathbf{i}\xi\eta)(\mathbf{i}\xi\zeta),$$

where $i$, $i'$ are a pair of equivalent symbols cogredient with $u$, and $p$, $p'$ are a pair of equivalent symbols cogredient with $x$.

3. Until van der Waerden succeeded in reducing six of the forms given by Ciamberlini it seems to have been regarded as obvious that the complete system of a number of ternary quadratics should be self-dual. In fact, the symbolic forms present themselves for consideration in reciprocal pairs, obtained from each other by interchanging $x$ with $u$ and $i$ with $\mathbf{i}$. It so happens that none of the processes of reduction actually carried out by earlier writers had ever involved the splitting up of a symbol $\mathbf{p}$ into its components $p, p'$; thus such reductions as were carried out applied equally to either of a pair of dual forms. But the forms of Ciamberlini reduced by van der Waerden are actually the duals of irreducible forms. Thus failure to reduce a form does not necessarily imply failure to reduce the dual, and we shall see below that many of the forms involving bracket factors such as $(\mathbf{pqr})$ or $(\mathbf{pq}x)$ are reducible when their duals are irreducible. It is therefore pertinent to remark that a reduction which does not involve the splitting up of a symbol $\mathbf{p}$ at any stage is equally applicable to the dual form, whereas if a symbol $\mathbf{p}$ is split up in the course of reduction the dual form needs separate consideration. It may be remarked, in parentheses, that it is the reductions of this latter type which are usually the most complicated.

It may happen that a form can be shown to reduce in such a way that in each term of the reduced expression one factor is an *invariant*; that is to say, that the variables $x$ and $u$ which occur in the form are never separated in the process of reduction. When this is the case, the same reduction will clearly demonstrate the reduction of the form in which $x, u$ are replaced by any cogredient symbols (or combinations of symbols) such that the form has an actual (as opposed to a symbolical) meaning. We shall make frequent use of this principle in what follows. A reduction of this type will be called an *invariant* reduction, and when it is necessary to emphasize the invariant nature of the reduction the equivalence sign will be replaced by the symbol $\equiv\!\!\!\!\equiv$.

It may also happen that a particular form whose symbolic expression involves a symbol $\mathbf{p}$ (or a symbol $p$ used as a suffix) can be shown to be reducible without making use of the fact that a symbolic product containing a factor $p_p$ is reducible, and without splitting $\mathbf{p}$ up into a pair of equivalent symbols $(pp')$. If this is the case we can clearly replace $\mathbf{p}$ by any cogredient combination of symbols. This process will also be used, on occasion, in the course of the work.

4. If $P$ is the symbolic expression of a form, all the forms whose symbolic expressions are obtained from $P$ by permuting the symbols referring to the different quadratics will be said to form a *set*, it being understood that the permutation applies to all the symbols referring to any given quadratic. For instance, the forms $\begin{smallmatrix} & 0 & & 2 & & u \\ x & & 1 & & 0 & \end{smallmatrix}$ and $\begin{smallmatrix} & 1 & & 3 & & u \\ x & & 2 & & 1 & \end{smallmatrix}$ belong to the same set, but $\begin{smallmatrix} & 1 & & 3 & & u \\ x & & 2 & & 0 & \end{smallmatrix}$ belongs to a different set. In enumerating the irreducible concomitants we shall give a representative of each set, and indicate the number of irreducible forms of the set which are of the same degrees as the representative form in the coefficients of the various quadratics; the total number of irreducible forms of the set in the system can then be written down. Thus, in the example given, the partial degrees are $3, 2, 1, 0, 0$; the set contains one irreducible form of these degrees and hence furnishes $5.4.3 = 60$ forms to the complete system.

Strictly speaking, we have no guarantee that the sets to be retained after our reductions have been carried out actually comprise irreducible forms until irreducibility has been demonstrated in the latter part of the paper. It will be convenient, however, to refer to the forms of these sets as 'irreducible', it being understood throughout that this term, at the present stage, means merely 'unreduced'.

In the case of the more complicated forms of the system, a set may well include a number of forms of the same partial degrees, of which certain linear combinations with constant coefficients are actually reducible, and a study of these combinations will be necessary in order to obtain the irreducible system. It will often happen that the symbolic expression $P$ of such a form is such that, on making a permutation $S$ of the symbols relating to certain of the quadratics, we obtain an expression $P'$ (representing another form of the same set as $P$) such that either $P-P'$ or $P+P'$ is reducible. Such a form $P$ will be said, in these two cases, to be symmetric or skew, respectively, with respect to the permutation $S$. If a form $P$ is symmetric with respect to two permutations $S_1$, $S_2$ it is symmetric with respect to every permutation of the group generated by $S_1$ and $S_2$; if it is symmetric with respect to $S_1$ and skew with respect to $S_2$ it is symmetric or skew with respect to an operation $S$ of the group generated by $S_1$ and $S_2$ according as the number of factors $S_2$ occurring in $S$ is even or odd. In particular, if the identical operation of the group can be expressed in terms of $S_1$ and $S_2$ in such a way as to involve an odd number of factors $S_2$, then $P$ is skew with respect to the identical operation and hence is reducible. Thus, for instance, if $P$ is symmetric for the interchange $(01)$ of symbols referring to the first and second quadratics, and skew for the interchange $(12)$ of symbols referring to the second and third, it must be reducible, for it is skew for the permutation $(01)(12) = (012)$ whose cube is the identity.

## Classification of possible forms

5. The possible types of symbolic product which arise may be classified as follows. If the product contains no bracket factors it consists entirely of factors such as $a_\beta$, $u_\beta$, $a_x$. By using the principle that the interchange of equivalent symbols produces an equivalent form, and that each symbol (other than $x$ or $u$) occurs just twice in a product which corresponds to an actual concomitant, it is easy to see that the only products of this type which can

actually represent irreducible forms can be represented in the contracted symbolic notation in one of the forms

$$\begin{array}{cccc} i & j & & i \\ p & q & \cdots & r \end{array}, \quad \begin{array}{ccccc} & i & j & & u \\ x & p & q & \cdots & r \end{array}, \quad \begin{array}{cccccc} & i & j & & k & \\ x & p & q & \cdots & r & x \end{array}, \quad \begin{array}{cccc} u & j & & u \\ p & q & \cdots & r \end{array},$$

in which (i) all the symbols $i, j, \ldots$ in the upper row are distinct and (ii) all the symbols in the lower row are distinct. Further, since a form containing a factor $a_\alpha$ is reducible, no pair of consecutive symbols, one from the upper row and one from the lower row, can refer to the same quadratic. If we call the grade of such a product the number of symbols $i, p, \ldots$ involved, then only forms of grade less than eleven need be considered, since otherwise, as only five quadratics are under consideration, a symbol would be repeated in either the upper or lower row. In fact, as we shall see, the maximum grade of an irreducible product is considerably lower. We observe that the same product can in certain cases be written in several ways, thus

$$\begin{array}{cccc} & i\; j & k & \\ x & p & q & x \end{array} = \begin{array}{cccc} & k\; j & i & \\ x & q & p & x \end{array} \quad \text{and} \quad \begin{array}{cccc} i & j & k & i \\ p & q & r & \end{array} = \begin{array}{cccc} i & k & j & i \\ r & q & p & \end{array} = \begin{array}{cccc} j & k & i & j \\ q & r & p & \end{array} = \text{etc.}$$

If a symbolic product contains bracket factors, it may be expressed in the form $PQ$, where $P$ is the product of the bracket factors and $Q$ the product of factors $a_\beta$, $u_\beta$, $a_x$. We have seen that it is only necessary to consider the cases in which all the factors of $P$ are either of type $(abc)$, $(abu)$ or of type $(\alpha\beta\gamma)$, $(\alpha\beta x)$. The two sets of forms which arise in this way are evidently dual to each other, and since in the process of discussing these products we shall deal simultaneously with a pair of dual forms, we may consider, as typical, the case in which $P$ is composed entirely of factors $(abc)$, $(abu)$. Since $P$ does not involve any symbol $\alpha$, any such symbol occurring in $Q$ must occur twice. Hence $Q$ can be expressed as the product of a number of expressions of one or other of the types

$$\begin{array}{cccc} i & j & & k \\ p & q & \cdots & r \end{array}, \quad \begin{array}{ccccc} i & j & & k & \\ p & q & \cdots & r & x \end{array}, \quad \begin{array}{cccc} u & j & & k \\ p & q & \cdots & r \end{array},$$

which we shall refer to, respectively, as chains, $x$-tags and $u$-tags. If the form is irreducible the factors occurring in these expressions are again subject to the restrictions that, in any chain or tag, all the upper symbols are distinct, all the lower symbols are distinct, and no two consecutive symbols, one upper and one lower, refer to the same quadratic.

We shall consider the forms in the following order:

A. Forms with no bracket factors.
B. Forms with bracket factors $(abu)$ but no bracket factors $(abc)$, and their duals.
C. Forms with a single bracket $(abc)$, and their duals.
D. Forms with two brackets of type $(abc)$, and their duals.
E. Forms with three brackets of type $(abc)$, and their duals.

It will be shown that no irreducible forms with more than three brackets of type $(abc)$ exist. As the number of possible forms in several of these systems is large, the systems will be further subdivided in due course.

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*The system of forms* A *with no bracket factors*

6. We take the forms of this set in ascending order of grade. The forms of the first four grades present no difficulty. They are all irreducible, and all the forms belonging to a set are linearly independent. Actually these forms refer to four or fewer quadratics, but we list them here as they are relevant to the complete system of types.

We shall distinguish the sets of forms of the irreducible system by numerals enclosed in square brackets, and after giving the symbolic form of a typical member of each set shall give its orders $(n_1, n_2)$ in the variables $x$ and $u$ and the partial degrees in the coefficients of the five fundamental quadratics, arranged as the symbol of a partition of the total degree of the form; and as stated in § 4, we shall indicate the number of irreducible forms of the set of given partial degrees. From forms of grade less than four we then obtain the following ten sets of forms:

*Grade* 1:

| | | | | |
|---|---|---|---|---|
| [1] | $0_x^2$ | Orders $(2, 0)$. | Degrees $\{1\}$. | One form. |
| [2] | $u_0^2$ | Orders $(0, 2)$. | Degrees $\{2\}$. | One form. |

*Grade* 2:

| | | | | |
|---|---|---|---|---|
| [3] | $\begin{smallmatrix} & 1 & & u \\ x & & 0 & \end{smallmatrix}$ | Orders $(1, 1)$. | Degrees $\{21\}$. | One form. |
| [4] | $0_0^2$ | Orders $(0, 0)$. | Degrees $\{3\}$. | One form. |
| [5] | $1_0^2$ | Orders $(0, 0)$. | Degrees $\{21\}$. | One form. |

*Grade* 3:

| | | | | |
|---|---|---|---|---|
| [6] | $\begin{smallmatrix} & 1 & & 2 & \\ x & & 0 & & x \end{smallmatrix}$ | Orders $(2, 0)$. | Degrees $\{21^2\}$. | One form. |
| [7] | $\begin{smallmatrix} u & & 2 & & u \\ & 0 & & 1 & \end{smallmatrix}$ | Orders $(0, 2)$. | Degrees $\{2^21\}$. | One form. |

*Grade* 4:

| | | | | |
|---|---|---|---|---|
| [8] | $\begin{smallmatrix} & 0 & & 2 & & u \\ x & & 1 & & 0 & \end{smallmatrix}$ | Orders $(1, 1)$ | Degrees $\{321\}$. | One form. |
| [9] | $\begin{smallmatrix} & 2 & & 3 & & u \\ x & & 0 & & 1 & \end{smallmatrix}$ | Orders $(1, 1)$. | Degrees $\{2^21^2\}$. | Four forms. |
| [10] | $\begin{smallmatrix} 2 & & 3 & & 2 \\ & 0 & & 1 & \end{smallmatrix}$ | Orders $(0, 0)$. | Degrees $\{2^21^2\}$. | One form. |

7. The possible forms of grade five are $\begin{smallmatrix} & i & & j & & k & \\ x & & p & & q & & x \end{smallmatrix}$ and the dual forms. Consider the former first. Since $p \neq q$ we may take as a typical form that in which $p = 0$ and $q = 1$. By squaring both sides of the identity

$$(ijk)\,(\mathbf{01}x) = \dot{\Sigma} i_0 j_1 k_x, \tag{7·1}$$

and neglecting terms containing invariant factors such as $i_0^2, j_1^2$ or $i_0 i_1 j_0 j_1$, it is easily seen that the sum of the six forms obtained from $\begin{smallmatrix} & i & & j & & k & \\ x & & p & & q & & x \end{smallmatrix}$ by permuting $i, j, k$ in all possible ways is reducible. Moreover, this reduction is an invariant reduction in the sense of § 3. If $i, j, k$ are all different from 0, 1, we thus see that there are five (and not six) irreducible forms of partial degrees $\{2^2 1^3\}$. If one of $i, j, k$ is 0 this can only be $k$, since otherwise the product contains the symbolic factor $0_0$ and reduces. Thus there are two forms of partial degrees $\{321^2\}$ in the corresponding set, and the identity (7·1) shows that the sum of these is reducible. A similar argument shows that all the forms are reducible if both 0 and 1 occur among $i, j, k$. Thus the irreducible forms left give the sets

[11] $\begin{smallmatrix} & 2 & & 3 & & 0 & \\ x & & 0 & & 1 & & x \end{smallmatrix}$ Orders (2, 0). Degrees $\{321^2\}$. One form.

[12] $\begin{smallmatrix} & 2 & & 3 & & 4 & \\ x & & 0 & & 1 & & x \end{smallmatrix}$ Orders (2, 0). Degrees $\{2^2 1^3\}$. Five forms.

8. We shall now show that the duals of all these forms are reducible. The reduction is elaborate, and, moreover, we shall find it convenient to obtain a generalized form of the reduction which will be applied in § 23 to reduce certain other forms. The principle of the method used here is to start with an expression which, being an actual product of forms, is clearly reducible, and to transform it into a certain linear expression in the forms which we are discussing (together with certain reducible terms). From the fact that this linear expression, and hence similar expressions obtained by permuting the symbols, is reducible, the reducibility of the forms under consideration is finally established. The selection of the starting point in a reduction of this type is a matter of trial and error, and the complexity of the reduction (which is one of the most elaborate to be considered) seems inherent in the nature of the case, as attempts to obtain a simpler reduction invariably led to failure.

We start from the relation, valid since each team is the product of two actual forms,

$$(pqr)\,(qri)\,(piu)\,.\,(q'sj)\,(q'p'j)\,(sp'u)+(pqs)\,(qsi)\,(piu)\,.\,(q'rj)\,(q'p'j)\,(rp'u)\equiv 0. \qquad (8\cdot1)$$

Here $p, p'$ and $q, q'$ are two sets of equivalent symbols. It will be noticed that the second term is derived from the first by interchanging $r$ and $s$. We shall proceed to transform the left-hand side of (8·1) into a form in which $p, p'$ have been replaced by a symbol $\mathbf{p}$ and $q, q'$ by $\mathbf{q}$. In the course of the work we shall have occasion to consider various terms, which will be denoted by $T_\lambda$ ($\lambda$ being a set of suffixes), and we make the convention that $T'_\lambda$ shall be the term obtained from $T_\lambda$ by interchanging $r$ and $s$.

By $[g_5, r \text{ of } g_2]$ the first term on the left of (8·1) becomes

$$(pqr)\,(piu)\,(q'sj)\,(sp'u)\,[(qq'i)\,(rp'j)+(qp'i)\,(q'rj)+(qji)\,(q'p'r)] = T_1+T_2+T_3, \text{ say,}$$

so that
$$T_1+T_2+T_3+T'_1+T'_2+T'_3\equiv 0. \qquad (8\cdot2)$$

Now
$$\begin{aligned} T_1 &= (pqr)\,(piu)\,(q'sj)\,(sp'u)\,(rp'j)\,i_q = \tfrac{1}{2}(piu)\,(sp'u)\,(rp'j)\,(\mathbf{q}\,rp\,sj)\,i_q \\ &= \tfrac{1}{2}(piu)\,(sp'u)\,(rp'j)\,[s_q(jrp)-j_q(srp)]\,i_q \\ &= T_{11}+T_{12}, \text{ say,} \end{aligned}$$

and
$$\begin{aligned} T_{11} &\equiv -\tfrac{1}{4}(\mathbf{p}\,jr\,iu)\,(\mathbf{p}\,jr\,us)\,s_q i_q \equiv \tfrac{1}{4}[(jiu)\,(rus)+(riu)\,(jus)]\,j_p r_p s_q i_q \\ &\equiv \tfrac{1}{2}(jiu)\,(rus)\,j_p r_p s_q i_q, \end{aligned}$$

by $[g_1, j \text{ of } g_2]$ applied to the second term. Hence

$$\begin{aligned} T_{11}+T'_{11} &\equiv \tfrac{1}{2}(jiu)\,(rus)\,j_p i_q[r_p s_q - s_p r_q] = \tfrac{1}{2}(sru)\,j_p i_q\,.\,(jiu)\,(rs\,\mathbf{p}\,\mathbf{q}) \\ &\equiv \tfrac{1}{2}(sru)\,j_p i_q[(jrs)\,i_p u_q+(irs)\,u_p j_q] \\ &= -\tfrac{1}{2}(sru)\,j_p i_q[(srj)\,i_p u_q+(sri)\,j_q u_p]. \end{aligned} \qquad (8\cdot3)$$

Again, $$T_{12}+T'_{12} = -\tfrac{1}{2}(srp)\,(piu)\,[(sp'u)\,(rp'j)-(rp'u)\,(sp'j)]\,i_q j_q. \tag{8·4}$$

Next, $$\begin{aligned} T_2+T'_2 &= (piu)\,(qp'i)\,(q'sj)\,(q'rj)\,[(pqr)\,(sp'u)+(pqs)\,(rp'u)] \\ &\equiv -(piu)\,(qp'i)\,(sr\,jq'\,pq)\,(sr\,jq'\,p'u) \\ &\equiv (piu)\,(qp'i)\,(srj)\,(srq')\,[(jpq)\,(q'p'u)+(q'pq)\,(jp'u)] \\ &= T^*_{21}+T^*_{22}, \text{ say,} \end{aligned}$$

where the asterisk indicates a form symmetrical in $r$ and $s$.

Transforming $T^*_{21}$ by $[g_2, j \text{ of } g_3]$, we find that

$$T^*_{21} \equiv (piu)\,(srq')\,(jpq)\,(q'p'u)\,[(jp'i)\,(srq)+(qp'j)\,(sri)] = T^*_{211}+T^*_{212}, \text{ say,}$$

and $$T^*_{211} \equiv -\tfrac{1}{2}(\mathbf{q}\;sr\,jp)\,(\mathbf{q}\;sr\,p'u)\,(piu)\,(jp'i), \tag{8·5}$$

while $$\begin{aligned} T^*_{212} &= -(pqj)\,(p'qj)\,(piu)\,(srq')\,(q'p'u)\,(sri) \\ &\equiv \tfrac{1}{2}(\mathbf{p}\;qj\,iu)\,(\mathbf{p}\;qj\,uq')\,(srq')\,(sri) \\ &\equiv -\tfrac{1}{2}[(qiu)\,(juq')+(jiu)\,(quq')]\,(srq')\,(sri)\,q_p j_p \\ &\equiv -(jiu)\,(quq')\,(srq')\,(sri)\,q_p j_p \end{aligned}$$

on applying $[g_1, j \text{ of } g_2]$ to the first term. Hence

$$\begin{aligned} T^*_{212} &\equiv (jiu)\,(srq')\,(sri)\,q_p j_p u_q = \tfrac{1}{2}(\mathbf{qp}\;sr)\,(jiu)\,(sri)\,j_p u_q \\ &\equiv \tfrac{1}{2}[j_q i_p(sru)+i_q u_p(srj)]\,(sri)\,j_p u_q. \end{aligned}$$

Thus $$T^*_{212} \equiv \tfrac{1}{2}(sri)\,(srj)\,i_q j_p u_p u_q+\tfrac{1}{2}(sri)\,(sru)\,i_p j_p j_q u_q. \tag{8·6}$$

And $$\begin{aligned} T^*_{22} &= (piu)\,(qp'i)\,(srj)\,(srq')\,(jp'u)\,p_q = \tfrac{1}{2}(piu)\,(srj)\,(jp'u)\,(\mathbf{q}\;p'i\,sr)\,p_q \\ &\equiv -\tfrac{1}{2}(piu)\,(srj)\,(jp'u)\,(sri)\,p_q p'_q \equiv \tfrac{1}{4}(\mathbf{pq}\;iu)\,(\mathbf{pq}\;uj)\,(srj)\,(sri) \\ &\equiv \tfrac{1}{4}(i_p j_q+i_q j_p)\,u_p u_q(srj)\,(sri). \end{aligned} \tag{8·7}$$

Next,

$$\begin{aligned} T_3 &= (piu)\,(sp'u)\,(qji)\,(q'p'r)\,.\,(pqr)\,(q'sj) = (piu)\,(sp'u)\,(qji)\,(q'p'r)\,[(pq'r)\,(qsj)+(\mathbf{q}\;rp\,sj)] \\ &= T_{31}+T_{32}, \text{ say;} \end{aligned}$$

and $$\begin{aligned} T_{31} &= -(piu)\,(sp'u)\,(qji)\,(qsj)\,(pq'r)\,(p'q'r) \equiv \tfrac{1}{2}(\mathbf{p}\;q'r\,iu)\,(\mathbf{p}\;q'r\,us)\,(qji)\,(qsj) \\ &\equiv -\tfrac{1}{2}[(q'iu)\,(rus)+(riu)\,(q'us)]\,(qji)\,(qsj)\,q'_p r_p \\ &\equiv -(q'iu)\,(rus)\,(qji)\,(qsj)\,q'_p r_p, \end{aligned}$$

by $[g_1, q' \text{ of } g_2]$ applied to the second term. Thus

$$\begin{aligned} T_{31}+T'_{31} &\equiv -(sru)\,(q'iu)\,(qji)\,q'_p[(qsj)\,r_p-(qrj)\,s_p] \\ &= -(sru)\,(q'iu)\,(qji)\,q'_p(sr\,jq\;\mathbf{p}) \\ &= -(sru)\,(q'iu)\,(qji)\,q'_p[(srq)\,j_p-(srj)\,q_p] \\ &= T^*_{311}+T^*_{312}, \text{ say,} \end{aligned}$$

the asterisk again denoting a form symmetrical in $r$ and $s$.

But, by $[g_3, u \text{ of } g_1]$,

$$T^*_{311} \equiv -(q'iu)\,(srq)\,q'_p j_p[(srj)\,(qui)+(sri)\,(qju)]$$
$$= (q'iu)\,(qiu)\,(srq)\,(srj)\,q'_p j_p - (q'iu)\,(sri)\,(srq')\,.\,(qju)\,q_p j_p - (q'iu)\,(sri)\,(qju) j_p (\mathbf{q}\,sr\,\mathbf{p})$$
$$\equiv -\tfrac{1}{2}(\mathbf{q}\,iu\,sr)(\mathbf{q}\,iu\,\mathbf{p})\,(srj) j_p - \tfrac{1}{2}(\mathbf{q}\,ju\,iu)\,(\mathbf{q}\,sr\,\mathbf{p})\,(sri) j_p$$
$$\equiv \tfrac{1}{2}(srj)\,[(sri)\,u_p + (sru)\,i_p]\,i_q j_p u_q - \tfrac{1}{2}(jiu)\,(\mathbf{q}\,sr\,\mathbf{p})\,(sri) j_p u_q$$
$$\equiv \tfrac{1}{2}(srj)\,[(sri)\,u_p + (sru)\,i_p]\,i_q j_p u_q + \tfrac{1}{2}(sri)\,[(srj)\,i_q u_p + (sru) j_q i_p] j_p u_q. \tag{8·8}$$

Also,
$$T^*_{312} \equiv -\tfrac{1}{2}(\mathbf{qp}\,iu)\,(\mathbf{qp}\,ji)\,(sru)\,(srj)$$
$$\equiv -\tfrac{1}{2}(srj)\,(sru)\,[j_p u_q + j_q u_p]\,i_p i_q. \tag{8·9}$$

Finally,
$$T_{32} = \tfrac{1}{2}(piu)\,(sp'u)\,(\mathbf{q}\,ji\,p'r)\,(\mathbf{q}\,rp\,sj) = \tfrac{1}{2}(piu)\,(sp'u)\,[(jrp)\,s_q - (srp) j_q]\,(\mathbf{q}\,ji\,p'r)$$
$$= T_{321} + T_{322}, \text{ say,}$$

and
$$T_{321} \equiv -\tfrac{1}{2}(piu)\,(sp'u)\,(jrp)\,(p'ji)\,r_q s_q,$$

so that
$$T_{321} + T'_{321} \equiv -\tfrac{1}{2}(piu)\,(p'ji)\,[(sp'u)\,(jrp) + (rp'u)\,(jsp)]\,r_q s_q$$
$$\equiv \tfrac{1}{2}(piu)\,(p'ji)\,(sr\,\mathbf{q}\,p'u)\,(sr\,\mathbf{q}\,pj)$$
$$= \tfrac{1}{2}(\mathbf{q}\,sr\,jp)\,(\mathbf{q}\,sr\,p'u)\,(piu)\,(jp'i)$$
$$\equiv -T^*_{211}, \tag{8·10}$$

while
$$T_{322} \equiv -\tfrac{1}{2}(piu)\,(sp'u)\,(srp)\,(jp'r)\,i_q j_q,$$

so that
$$T_{322} + T'_{322} \equiv -\tfrac{1}{2}(srp)\,(piu)\,[(sp'u)\,(jp'r) - (rp'u)\,(jp's)]\,i_q j_q$$
$$\equiv -(T_{12} + T'_{12}). \tag{8·11}$$

From the relations (8·2) to (8·11), we obtain, after a little reduction, and after multiplication by 4,

$$(sri)\,(srj)\,[i_p j_q + 7 i_q j_p]\,u_p u_q + (sri)\,(sru)\,[4 i_p u_q - 2 i_q u_p] j_p j_q$$
$$+ (srj)\,(sru)\,[-2 j_p u_q - 2 j_q u_p]\,i_p i_q \equiv 0. \tag{8·12}$$

The unsymmetrical nature of the left-hand member of (8·12) with respect to $i$, $j$ and $p$, $q$ derives from the unsymmetrical nature of the left-hand member of (8·1), and forms the basis for our reduction.

We now suppose that $s$, which hitherto has been arbitrary, is a symbol equivalent to $r$, and replace $(rs)$ by $\mathbf{r}$ in (8·12). Each symbolic product on the left of (8·12) is then of the form $(u_\xi i_\eta j_\zeta u)$, where $\xi$, $\eta$, $\zeta$ are $p$, $q$, $r$ in some order. Denoting this form for the moment by the symbol $[\xi\eta\zeta]$, we see that (8·12) becomes

$$[prq] + 7[qrp] + 4[rpq] - 2[rqp] - 2[qpr] - 2[pqr] \equiv 0. \tag{8·13}$$

By permuting $p$, $q$, $r$ in all possible ways we obtain six linear relations of the form (8·13). It is easily seen that these relations imply that $[pqr] \equiv 0$. Hence the form $\begin{smallmatrix} u & & i & & j & & u \\ & p & & q & & r & \end{smallmatrix}$ is reducible, as was to be proved.

9. The forms of grade six in the system A are of two kinds, invariants and bilinear forms. The invariants are typified by $\begin{smallmatrix} i & & j & & k & & i \\ & p & & q & & r & \end{smallmatrix}$. Since we are considering at most five quadratics, at

least one symbol is common to the upper and lower rows; and if the form is to be irreducible it must not occur in adjacent positions. We suppose, then, that $q = i$, which is permissible since the form can be written in such a way that any one of $i, j, k$ is at the two ends on the upper row. Now, by the invariant reduction of § 7, the form $\begin{smallmatrix} & j & & k & & i & \\ x & & i & & r & & x \end{smallmatrix}$ is skew in $j$ and $k$. We may replace $x$ by $p$ in the invariant reduction, and obtain the relation

$$\begin{smallmatrix} i & & j & & k & & i \\ & p & & i & & r & \end{smallmatrix} + \begin{smallmatrix} i & & k & & j & & i \\ & p & & i & & r & \end{smallmatrix} \equiv 0.$$

Dually,

$$\begin{smallmatrix} i & & j & & k & & i \\ & p & & i & & r & \end{smallmatrix} + \begin{smallmatrix} i & & j & & k & & i \\ & r & & i & & p & \end{smallmatrix} \equiv 0.$$

Thus, if $j, k, p, r$ are all distinct there is just one independent invariant of degrees 3, 2, 2, 1, 1 in assigned quadratics; and there are no irreducible invariants if $j, k, p, r$ are not all different. Hence we have the single set of forms (Turnbull 1910)

[13] $\begin{smallmatrix} 0 & & 3 & & 4 & & 0 \\ & 1 & & 0 & & 2 & \end{smallmatrix}$ Orders $(0, 0)$. Degrees $\{32^2 1^2\}$. One form.

10. The other forms of grade six are bilinear forms of the type $\begin{smallmatrix} & i & & j & & k & & u \\ x & & p & & q & & r & \end{smallmatrix}$. These are in fact all reducible, but the analysis which demonstrates this is somewhat lengthy. Just as in § 9, there is at least one symbol common to the upper and lower rows in the product. We denote this symbol by $i$, the remaining upper symbols by $j$, $k$ and the remaining lower symbols by $p$, $q$. We have, for an irreducible form, $j, k, p, q \neq i$, $j \neq k$, $p \neq q$, but there are forms of this kind not obviously reducible with, e.g. $p = k$, and we make, at present, no hypothesis as to the inequality of such symbols.

We first enumerate the possible forms of this type with $ijk$ as upper symbols and $ipq$ as lower symbols, omitting these which are clearly reducible on account of the presence of a factor $i_i$. These forms are sixteen in number, namely

$$L_1 = \begin{smallmatrix} & i & & j & & k & & u \\ x & & p & & i & & q & \end{smallmatrix}, \quad M_1 = \begin{smallmatrix} & i & & j & & k & & u \\ x & & p & & q & & i & \end{smallmatrix}, \quad N_1 = \begin{smallmatrix} & j & & i & & k & & u \\ x & & p & & q & & i & \end{smallmatrix}, \quad P_1 = \begin{smallmatrix} & j & & k & & i & & u \\ x & & i & & p & & q & \end{smallmatrix},$$

and three other sets of forms, which will be denoted by similar symbols bearing suffixes 2, 3, 4, obtained from those just written by interchanging, in the respective cases, $j, k$; $p, q$; $j, k$ and $p, q$. We proceed to deduce a number of relations between these forms, the first two sets—(10·1) and (10·2) below—of which are due to Turnbull.

The first set of relations arises by polarizing the equivalences

$$\begin{smallmatrix} & i & & j & & k & \\ x & & p & & i & & x \end{smallmatrix} + \begin{smallmatrix} & i & & k & & j & \\ x & & p & & i & & x \end{smallmatrix} \equiv 0, \quad \begin{smallmatrix} u & & i & & j & & u \\ & p & & q & & i & \end{smallmatrix} + \begin{smallmatrix} u & & i & & j & & u \\ & q & & p & & i & \end{smallmatrix} \equiv 0,$$

obtained in § 7, with respect to $u_q$ and $k_x$, and interchanging $p$, $q$ and $j$, $k$. It is easily verified that this gives

$$\left.\begin{aligned} L_1 + L_2 + P_1 + P_2 \equiv 0, \quad N_1 + N_3 + P_1 + P_3 \equiv 0, \\ L_3 + L_4 + P_3 + P_4 \equiv 0, \quad N_2 + N_4 + P_2 + P_4 \equiv 0. \end{aligned}\right\} \tag{10·1}$$

The second set is derived from the identity

$$\dot{\Sigma} i_x j_i k_p u_q = 0$$

(the dot indicating a determinantal permutation of the suffixes) by multiplying by a factor of the form $i_p j_i k_q$ (so that the product leads to an actual form) and expanding the resulting determinant, neglecting terms which represent products of actual forms. The relations so arising are found, without difficulty, to be

$$\left.\begin{aligned} L_1+M_2+N_4+P_3 \equiv 0, \quad L_3+M_4+N_2+P_1 \equiv 0,\\ L_2+M_1+N_3+P_4 \equiv 0, \quad L_4+M_3+N_1+P_2 \equiv 0,\end{aligned}\right\} \tag{10·2}$$

the reductions being invariant ones.

Another relation (not, now, an invariant reduction) arises from the equivalence

$$0 \equiv (\mathbf{ip}x)\begin{pmatrix} & j & \\ i & & p\end{pmatrix}.(iku)\begin{pmatrix} i & & k\\ & q & \end{pmatrix} \equiv \begin{pmatrix} & j & \\ i & & p\end{pmatrix}\begin{pmatrix} i & & k\\ & q & \end{pmatrix}[i_p k_x u_i + i_x k_i u_p - i_x k_p u_i]$$
$$= N_4+L_4-M_4.$$

By interchanging $j$, $k$ and $p$, $q$ we obtain

$$L_1-M_1+N_1 \equiv 0, \quad L_2-M_2+N_2 \equiv 0, \quad L_3-M_3+N_3 \equiv 0, \quad L_4-M_4+N_4 \equiv 0. \tag{10·3}$$

Though this is not an invariant reduction it has the property, shared by (10·1) and (10·2), that it does not involve the splitting of a symbol **p** or **q** into $(pp')$ or $(qq')$; thus (10·1), (10·2) and (10·3) remain valid if **p** is replaced, for instance, by $(kl)$, where $k$, $l$ are symbols cogredient with $u$. The next reduction to be obtained does not possess this property, but is an invariant reduction.

We have
$$0 \equiv (ijk)(ijp)k_q p_q . p'_x p'_i u_i \equiv [(ip'k)j_i + (ijp')k_i](ijp)k_q p_q p'_x u_i$$
$$= T_1+T_2, \text{ say.}$$

Now, by $[g_2, u_i]$ followed by $[g_1, j_i]$,

$$\begin{aligned} T_1 &\equiv (ip'k)(iju)j_i k_q p_q p'_x p_i \equiv (ijk)(iju)p'_i k_q p_q p'_x p_i\\ &\equiv -\tfrac{1}{2}(\mathbf{piq})(\mathbf{pi}x)(ijk)(iju)k_q = -\tfrac{1}{2}(\mathbf{piq})(ijk).(\mathbf{pi}x)(iju).k_q\\ &\equiv -\tfrac{1}{2}[-i_p j_q k_i + i_q j_p k_i - i_q j_i k_p][-i_p j_x u_i + i_x j_p u_i - i_x j_i u_p]k_q\\ &\equiv -\tfrac{1}{2}[-i_p j_q k_i . i_x j_p u_i - i_q j_p k_i . i_p j_x u_i - i_q j_p k_i . i_x j_i u_p - i_q j_i k_p . i_x j_p u_i]k_q\\ &= \tfrac{1}{2}(M_1+N_1+L_4+M_4),\end{aligned}$$

and
$$T_2 \equiv -\tfrac{1}{2}(\mathbf{p}\,ij\,\mathbf{q})(\mathbf{p}\,ij\,x)k_q k_i u_i \equiv \tfrac{1}{2}(i_q j_x + i_x j_q)i_p j_p k_q k_i u_i = \tfrac{1}{2}(N_1+M_1).$$

Thus, adding and multiplying by two, and exchanging $j$, $k$ and $p$, $q$, we find the relations

$$\left.\begin{aligned} L_1+M_1+2(M_4+N_4) \equiv 0, \quad L_3+M_3+2(M_2+N_2) \equiv 0,\\ L_2+M_2+2(M_3+N_3) \equiv 0, \quad L_4+M_4+2(M_1+N_1) \equiv 0.\end{aligned}\right\} \tag{10·4}$$

It is easily verified that the invariant reductions (10·1), (10·2), (10·4) imply the relations

$$\left.\begin{aligned} &M_1+M_2+N_1+N_2 \equiv 0, && M_3+M_4+N_3+N_4 \equiv 0,\\ &-L_1 \equiv M_1+2M_4+2N_4, && P_1 \equiv 2M_2+M_3-M_4+N_2,\\ &-L_2 \equiv M_2+2M_3+2N_3, && P_2 \equiv 2M_1-M_3+M_4+N_1,\\ &-L_3 \equiv 2M_2+M_3+2N_2, && P_3 \equiv M_1-M_2+2M_4+N_4,\\ &-L_4 \equiv 2M_1+M_4+2N_1, && P_4 \equiv -M_1+M_2+2M_3+N_3,\end{aligned}\right\} \tag{10·5}$$

which accordingly remain valid if $x$ and $u$ are replaced by any cogredient expressions $\xi$, $\omega$ such that $\omega_\xi$ has an actual meaning; and that the reductions (10·1), (10·2) and (10·3), which hold independently of the nature of $\mathbf{p}$ and $\mathbf{q}$, imply

$$\left.\begin{aligned} L_1+L_2 \equiv L_3+L_4 \equiv N_1+N_3 \equiv N_2+N_4 \equiv 0, \\ P_n \equiv 0, \quad M_n \equiv L_n+N_n \quad (n=1,2,3,4); \end{aligned}\right\} \tag{10·6}$$

these remaining valid, for instance, if $\mathbf{p}$ is replaced by $(lm)$. For the forms immediately under discussion both sets of equivalences are valid. From (10·6) and (10·4) we deduce, without difficulty, that

$$\left.\begin{aligned} N_1 \equiv -N_2 \equiv -N_3 \equiv N_4 \equiv \nu, \\ L_1 \equiv -L_2 \equiv \lambda_1, \quad L_3 \equiv -L_4 \equiv \lambda_3, \\ 2(\lambda_1-\lambda_3)+5\nu \equiv 0. \end{aligned}\right\} \tag{10·7}$$

Consider now the expressions

$$S_1 = (ijp')\,(kpu)\,k_i p'_q i_q j_i p_x, \quad S_2 = (ikp')\,(jpu)\,j_i p'_q i_q k_i p_x.$$

Then

$$\begin{aligned} S_1 - S_2 &= [(ijp')\,(kpu) - (ikp')\,(jpu)]\,j_i k_i i_q p'_q p_x \\ &= [(ipp')\,(kju) + (iup')\,(kpj)]\,j_i k_i i_q p'_q p_x \equiv 0, \end{aligned}$$

so that $S_1$ is symmetric in $j$ and $k$. But

$$\begin{aligned} 0 &\equiv (ijk)\,(ijp)\,(kpu)\,.\,(\mathbf{iq}x)\,p'_i p'_q \\ &\equiv (ijk)\,(kpu)\,[i_q j_x p_i - i_q j_i p_x + i_x j_i p_q - i_x j_q p_i]\,p'_i p'_q \\ &= T_1+T_2+T_3+T_4, \text{ say.} \end{aligned}$$

Now

$$\begin{aligned} T_1 &\equiv -\tfrac{1}{2}(\mathbf{piq})\,(\mathbf{pi}\,uk)\,(ijk)\,i_q j_x \equiv -\tfrac{1}{2} i_p (j_i k_q - j_q k_i)\,(u_p k_i - u_i k_p)\,i_q j_x \\ &\equiv -\tfrac{1}{2}[i_p j_i k_q u_p k_i i_q j_x + i_p j_q k_i u_i k_p i_q j_x] = -\tfrac{1}{2}(P_3+N_3), \\ T_2 &\equiv -S_1, \text{ by } [g_1, p'_i], \\ T_3 &\equiv -\tfrac{1}{2}(\mathbf{pq}\,uk)\,(\mathbf{pqi})\,(ijk)\,i_x j_i \equiv -\tfrac{1}{2}(u_p k_q - u_q k_p)\,(i_p j_q - i_q j_p)\,k_i i_x j_i \\ &\equiv \tfrac{1}{2}[u_p k_q i_q j_p + u_q k_p i_p j_q]\,k_i i_x j_i = \tfrac{1}{2}(L_4+L_2), \\ T_4 &\equiv \tfrac{1}{2}(\mathbf{pi}\,uk)\,(\mathbf{piq})\,(ijk)\,i_x j_q \equiv \tfrac{1}{2}(u_p k_i - u_i k_p)\,(i_q j_p k_i - i_q j_i k_p + i_p j_i k_q)\,i_x j_q \\ &\equiv \tfrac{1}{2}[-u_p k_i i_q j_i k_p - u_i k_p i_q j_p k_i - u_i k_p i_p j_i k_q]\,i_x j_q \\ &= -\tfrac{1}{2}(L_3+M_3+M_2). \end{aligned}$$

Thus, as

$$T_1+T_2+T_3+T_4 \equiv 0,$$
$$2S_1 \equiv L_2 - L_3 + L_4 - M_2 - M_3 - N_3 - P_3.$$

But $S_1$ is symmetric in $j$ and $k$. Hence

$$L_2 - L_3 + L_4 - M_2 - M_3 - N_3 - P_3 \equiv L_1 - L_4 + L_3 - M_1 - M_4 - N_4 - P_4,$$

and so, eliminating $M_n$ and $P_n$ by (10·6),

$$3(L_3 - L_4) \equiv (N_1 - N_2) - 2(N_3 - N_4).$$

By interchanging $p$ and $q$ we obtain, similarly,

$$3(L_1 - L_2) \equiv -2(N_1 - N_2) + (N_3 - N_4).$$

Using (10·7) we then find

$$6\lambda_3 \equiv 6\nu \equiv -6\lambda_1, \quad 2(\lambda_1 - \lambda_3) + 5\nu \equiv 0,$$

whence $\lambda_1 \equiv \lambda_3 \equiv \nu \equiv 0$. From (10·6) and (10·7) it now follows that all the sixteen forms $L_n$, $M_n$, $N_n$, $P_n$ are reducible.


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11. The reduction of these bilinear forms is not an invariant reduction, but it can easily be shown that those forms of the set expressible in the form $\begin{smallmatrix} & i & & j & & k & & u \\ x & & p & & i & & j & \end{smallmatrix}$ admit an invariant reduction, so that $\begin{smallmatrix} & i & & j & & k & & \omega \\ \xi & & p & & i & & j & \end{smallmatrix} \equiv 0$ whenever $\omega_\xi$ is the symbolic expression of an actual form. For if, in the notation of § 10, $q=j$, the forms $L_2$, $L_3$, $M_1$, $M_2$, $M_3$, $N_2$, $N_3$, $P_4$ are all $\equiv 0$ on account of the presence of the factor $\begin{smallmatrix} j & \\ & j \end{smallmatrix}$, and from (10·5) it follows that $N_1 \equiv 0$, $L_1 \equiv 0$.

It follows from this that any form of the set A which, in the contracted notation, contains symbols of the form $\dots \begin{smallmatrix} i & & j & & k & \\ & p & & i & & j \end{smallmatrix} \dots$ or $\dots \begin{smallmatrix} & k & & j & & i \\ j & & i & & p & \end{smallmatrix} \dots$ (which is the same thing) is $\equiv 0$.

12. The forms of grade seven included in the system A comprise the forms $\begin{smallmatrix} & i & & j & & k & & l & \\ x & & p & & q & & r & & x \end{smallmatrix}$ and their duals. These are all reducible, in fact, it will be shown that

$$\begin{smallmatrix} & i & & j & & k & & l & \\ \xi & & p & & q & & r & & \eta \end{smallmatrix} \equiv 0 \tag{12·1}$$

for arbitrary $\xi$, $\eta$ such that the expression on the left has an actual meaning.

At least two symbols occur in both the top and bottom rows. We denote these by $i, j$, the remaining symbols in the upper row by $k$, $l$ and the remaining symbol in the lower row by $p$. Neglecting expressions of the form (12·1) which are reducible on account of a factor of the form $\begin{smallmatrix} i & \\ & i \end{smallmatrix}$ and forms containing $\dots \begin{smallmatrix} i & & j & & k & \\ & p & & i & & j \end{smallmatrix} \dots$ which were shown to be reducible in § 11, there remain for consideration 32 expressions, namely, the eight forms

$$A_1 = \begin{smallmatrix} & j & & k & & i & & l & \\ \xi & & i & & j & & p & & \eta \end{smallmatrix}, \quad B_1 = \begin{smallmatrix} & i & & k & & j & & l & \\ \xi & & j & & i & & p & & \eta \end{smallmatrix}, \quad C_1 = \begin{smallmatrix} & j & & k & & l & & i & \\ \xi & & i & & j & & p & & \eta \end{smallmatrix}, \quad D_1 = \begin{smallmatrix} & i & & k & & l & & j & \\ \xi & & j & & i & & p & & \eta \end{smallmatrix},$$

$$E_1 = \begin{smallmatrix} & j & & k & & i & & l & \\ \xi & & i & & p & & j & & \eta \end{smallmatrix}, \quad F_1 = \begin{smallmatrix} & i & & k & & j & & l & \\ \xi & & j & & p & & i & & \eta \end{smallmatrix}, \quad G_1 = \begin{smallmatrix} & j & & k & & l & & i & \\ \xi & & i & & p & & j & & \eta \end{smallmatrix}, \quad H_1 = \begin{smallmatrix} & k & & j & & i & & l & \\ \xi & & i & & p & & j & & \eta \end{smallmatrix};$$

eight forms $A_2, \dots, H_2$ obtained by interchanging $k$ and $l$, and sixteen further forms $A'_1, \dots, H'_2$ derived from these by interchanging $\xi$ and $\eta$. It will be noticed that the interchange of $i$ and $j$ interchanges these forms in pairs $(A_n, B_n)$, $(A'_n, B'_n)$, $(C_n, D_n)$, $(C'_n, D'_n)$, $(E_n, F_n)$, $(E'_n, F'_n)$, $(G_n, G'_m)$, $(H_n, H'_m)$, where $n, m = 1, 2$ and $n \neq m$.

In the forms of § 10 replace $x$ by $\xi$, $j$ by $l$, $q$ by $j$, and $u$ by $jj_\eta$. Then the forms containing the factor $u_q$ clearly reduce, and we have in fact

$$L_1 \equiv 0, \quad L_2 \equiv 0, \quad L_3 = D_2, \quad L_4 = D_1; \qquad M_1 = C'_1, \quad M_2 = C'_2, \quad M_3 = G'_1, \quad M_4 = G'_2;$$
$$N_1 = A'_1, \quad N_2 = A'_2, \quad N_3 = E'_1, \quad N_4 = E'_2; \qquad P_1 \equiv 0, \quad P_2 \equiv 0, \quad P_3 \equiv 0, \quad P_4 \equiv 0.$$

Now these forms satisfy the conditions for the invariant reduction (10·5). These relations, together with the fact that $L_1$, $L_2$, $P_n$ are all $\equiv 0$, imply

$$N_1 \equiv -N_2 \equiv -N_3 \equiv N_4 \equiv M_1 \equiv -M_2 = \lambda, \quad \text{say}, \quad M_3 \equiv -M_4 \equiv \tfrac{3}{2}\lambda, \quad L_3 \equiv -L_4 \equiv \tfrac{5}{2}\lambda.$$

Hence we have the relations

$$A'_1 \equiv -A'_2 \equiv -E'_1 \equiv E'_2 \equiv C'_1 \equiv -C'_2 \equiv \lambda, \quad G'_1 \equiv -G'_2 \equiv \tfrac{3}{2}\lambda, \quad D_2 \equiv -D_1 \equiv \tfrac{5}{2}\lambda,$$

and, in particular, $D_2 \equiv \frac{5}{2}C'_1$. By simultaneous interchange of $ij$, $kl$, $\xi\eta$, we obtain, similarly, $C'_1 \equiv \frac{5}{2}D_2$. Hence $\lambda \equiv 0$. Thus, permuting $kl$ and $\xi\eta$, we see that all the forms except those of the set $H$ are $\equiv 0$.

For these last forms, put $q = j$, $x = \xi$, $u = ll_\eta$ in the forms of § 10. Then $L_1 \equiv 0$ on account of the factors $\begin{smallmatrix} i & j & & k & \\ & p & \cdot & i & j \end{smallmatrix}$ and $M_1 \equiv 0$ on account of the factor $\begin{smallmatrix} j & \\ & \cdot\cdot \\ j & \end{smallmatrix}$. Hence, by (10·4), which applies here, $M_4 + N_4 \equiv 0$. But $M_4 = F_1$ and $N_4 = H'_2$. Hence $H'_2 \equiv -F_1 \equiv 0$, and by interchange of $k$, $l$ or $\xi$, $\eta$ we see that the remaining forms of the system are $\equiv 0$. In particular, if $\xi = \eta = x$, we see that all the possible forms admit an invariant reduction. The dual forms reduce in exactly the same way.

13. It is now a simple matter to show that the remaining forms of the system A, of grade eight or more, are reducible (and in fact admit an invariant reduction). For any such form may be obtained by substituting appropriate symbols or combinations of symbols for $\xi$ and $\eta$ in the form on the left of (12·1), and hence all such forms are $\equiv 0$. The enumeration of the irreducible forms belonging to the system A is accordingly complete.

## *Subdivision of the system of forms* B

14. The forms of the system B occur in dual pairs, and we shall consider the two forms of such a pair together. Any form of the system can be expressed as a product $PQ$, where $P$ is a product of factors of the form $(iju)$, or the dual of such a product, and $Q$ is made up of chains and tags. We shall take as the representative of a pair of dual forms that in which $P$ is composed of factors $(iju)$, and subdivide the system according to the nature of this product. It is easy to see that if the form does not include a reducible factor, the *a priori* possibilities, for five or fewer quadratics, give rise to the following subsystems, in the first four of which no product $Q$ can occur, as the product $P$ represents an actual form:

| | | | |
|---|---|---|---|
| $B_1$ | $(iju)^2$, | $B_6$ | $(iju)\,(jku)\,Q$, |
| $B_2$ | $(iju)\,(jku)\,(kiu)$, | $B_7$ | $(iju)\,(jku)\,(klu)\,Q$, |
| $B_3$ | $(iju)\,(jku)\,(klu)\,(liu)$, | $B_8$ | $(iju)\,(jku)\,(klu)\,(lmu)\,Q$, |
| $B_4$ | $(iju)\,(jku)\,(klu)\,(lmu)\,(miu)$, | $B_9$ | $(iju)\,(klu)\,Q$, |
| $B_5$ | $(iju)\,Q$, | $B_{10}$ | $(iju)\,(klu)\,(lmu)\,Q$. |

### *The subsystems* $B_1$, $B_2$, $B_3$, $B_4$

15. These forms present no difficulty. The first two give rise to the irreducible sets:

| | | | | |
|---|---|---|---|---|
| [14] | $(01u)^2$ | Orders $(0, 2)$. | Degrees $\{1^2\}$. | One form. |
| [15] | $(\mathbf{01}x)^2$ | Orders $(2, 0)$. | Degrees $\{2^2\}$. | One form. |
| [16] | $(01u)\,(12u)\,(20u)$ | Orders $(0, 3)$. | Degrees $\{1^3\}$. | One form. |
| [17] | $(\mathbf{01}x)\,(\mathbf{12}x)\,(\mathbf{20}x)$ | Orders $(3, 0)$. | Degrees $\{2^3\}$. | One form. |

The other forms are reducible (Turnbull 1910). For the forms $B_3$ this follows at once on squaring the identity
$$(iju)\,(klu)-(jku)\,(liu) = (iku)\,(jlu),$$
giving
$$(iju)\,(jku)\,(klu)\,(liu) = \tfrac{1}{2}[(iju)^2\,(klu)^2+(jku)^2\,(liu)^2-(iku)^2\,(jlu)^2]. \qquad (15\cdot1)$$

Operating on this identity with the symbolic differential operator $m(\partial/\partial i)$ and multiplying by $(miu)$ we obtain
$$(iju)\,(jku)\,(klu)\,(lmu)\,(miu)-(imu)\,(mju)\,(jku)\,(klu)\,(liu) \equiv 0. \qquad (15\cdot2)$$

But
$$\begin{aligned}(iju)\,(jku)\,(klu)\,(lmu)\,(miu)+(imu)\,(mju)\,(jku)\,(klu)\,(liu)\\ = (imu)\,(jku)\,(klu)\,[-(iju)\,(lmu)+(mju)\,(liu)]\\ = -(imu)^2.(jku)\,(klu)\,(lju) \equiv 0,\end{aligned} \qquad (15\cdot3)$$

and hence, from (15·2) and (15·3), $(iju)\,(jku)\,(klu)\,(lmu)\,(miu) \equiv 0$, so that the forms $B_4$ are reducible. The same reasoning applies to the dual forms.

*The subsystem $B_5$*

16. These are the forms $(iju)\,Q$, and two cases arise according as $Q$ is a chain $(i,j)$ with $i$ and $j$ as the extreme links in the upper row, or the product $(i)\,(j)$ of two tags, one commencing with $i$ and the other with $j$.

If $(i,j)$ is a chain the possibilities which have to be considered are

$$\text{(i)}\ \ Q=\begin{pmatrix} i & & j \\ & p & \end{pmatrix},\quad \text{(ii)}\ \ Q=\begin{pmatrix} i & & k & & j \\ & p & & q & \end{pmatrix},\quad \text{(iii)}\ \ Q=\begin{pmatrix} i & & k & & l & & j \\ & p & & q & & r & \end{pmatrix},\quad \text{(iv)}\ \ Q=\begin{pmatrix} i & & k & & l & & m & & j \\ & p & & q & & r & & s & \end{pmatrix}.$$

The first two possibilities yield no reductions, and give rise to the irreducible sets

| | | | | |
|---|---|---|---|---|
| [18] | $(12u)\begin{pmatrix} 1 & & 2 \\ & 0 & \end{pmatrix}$ | Orders (0, 1). | Degrees $\{21^2\}$. | One form. |
| [19] | $(\mathbf{01}x)\begin{pmatrix} & 2 & \\ 0 & & 1 \end{pmatrix}$ | Orders (1, 0). | Degrees $\{2^21\}$. | One form. |
| [20] | $(01u)\begin{pmatrix} 0 & & 2 & & 1 \\ & 1 & & 0 & \end{pmatrix}$ | Orders (0, 1). | Degrees $\{3^21\}$. | One form. |
| [21] | $(\mathbf{01}x)\begin{pmatrix} & 1 & & 0 & \\ 0 & & 2 & & 1 \end{pmatrix}$ | Orders (1, 0). | Degrees $\{3^22\}$. | One form. |
| [22] | $(02u)\begin{pmatrix} 0 & & 3 & & 2 \\ & 1 & & 0 & \end{pmatrix}$ | Orders (0, 1). | Degrees $\{321^2\}$. | Two forms. |
| [23] | $(\mathbf{01}x)\begin{pmatrix} & 3 & & 0 & \\ 0 & & 2 & & 1 \end{pmatrix}$ | Orders (1, 0). | Degrees $\{32^21\}$. | Two forms. |
| [24] | $(23u)\begin{pmatrix} 2 & & 4 & & 3 \\ & 0 & & 1 & \end{pmatrix}$ | Orders (0, 1). | Degrees $\{2^21^3\}$. | Six forms. |
| [25] | $(\mathbf{01}x)\begin{pmatrix} & 3 & & 4 & \\ 0 & & 2 & & 1 \end{pmatrix}$ | Orders (1, 0). | Degrees $\{2^31^2\}$. | Six forms. |

17. The next system to be considered is $(iju)\begin{matrix} i & & k & & l & & j \\ & p & & q & & r & \end{matrix}$. All these forms are in fact reducible. The argument which establishes this is very similar to that of § 12. The two rows in the chain must contain at least two common symbols, so that we may take $ijkl$, $ijp$ to denote the symbols which occur, in some order, in these rows. The bracket outside is $(abu)$, where $a, b$ are the extreme symbols in the upper row of the chain. By replacing $\xi$ by $(lu)$ and $\omega$ by $l$ in the form $\begin{matrix} & i & & j & & k & & \omega \\ \xi & & p & & i & & j & \end{matrix}$, which $\equiv 0$ by § 11, we see that $(ilu)\begin{pmatrix} i & & j & & k & & l \\ & p & & i & & j & \end{pmatrix} \equiv 0$, which can equally be written in the form $-(liu)\begin{pmatrix} l & & k & & j & & i \\ & j & & i & & p & \end{pmatrix} \equiv 0$. Neglecting these forms, the possible forms to be considered are obtained from those enumerated in § 12 by merely suppressing the extreme factors $a_\xi$, $b_\eta$ and replacing them by $(abu)$. The effect of this is that the forms denoted by accented symbols do not arise for separate consideration, since they arise from the corresponding unaccented forms by a change of sign. We may denote the forms so arising by the same symbols as those used in § 12, remembering that now we have the additional relations $S+S' = 0$, where $S$ is any one of the sixteen forms $A_1, \ldots, H_2$.

The reduction of these follows, *mutatis mutandis*, the same course as that in § 12. In the forms of § 10 we replace $x, j, q, u$ by $(ju), l, j, j$ respectively. The identification of the resulting forms is then exactly that of § 12, and the argument of that section shows that the forms $A_n, \ldots, G_n$ are all reducible. The reducibility of the forms $H_n$ follows in the same way by replacing $x, q, u$ by $(lu), j, l$ in the forms of § 10. The same argument applies to the dual forms.

The remaining system of this kind has for typical form $(iju)\begin{pmatrix} i & & k & & l & & m & & j \\ & p & & q & & r & & s & \end{pmatrix}$, which is seen to be reducible on writing $(ju)$ for $x$ and $j$ for $u$ in the relation $\begin{pmatrix} & i & & k & & l & & m & & u \\ x & & p & & q & & r & & s & \end{pmatrix} \equiv 0$, proved in § 13. The dual form reduces in the same way.

18. The other forms $B_5$ can be written $(iju)(i)(j)$, where $(i)$ and $(j)$ denote tags. If a symbol $k$ (or $p$) occurs in the top (or bottom) rows of both $(i)$ and $(j)$, the product $(i)(j)$ can be rearranged so as to contain a chain $(i, j)$ as a factor, and the form reduces. We may therefore suppose that no such symbol occurs. Since $(iju) = -(jiu)$ we may suppose, without loss of generality, that the tag $(j)$ is not longer than the tag $(i)$. It is convenient to consider the forms in decreasing order of length of $(i)$.

In the first place, all the forms in which three (or more) symbols of type $p$ occur as lower symbols in $(i)$ are reducible (Turnbull, 1910). For any such form may be written

$$(iju)\begin{pmatrix} i & & k & & l & & \omega \\ & p & & q & & r & \end{pmatrix}(j),$$

where $\omega$ stands for $u$ or $\begin{pmatrix} m & \\ & x \end{pmatrix}$ or $\begin{pmatrix} m & & u \\ & s & \end{pmatrix}$. By $[g_1, k_q]$ we see that this form it symmetric in $i, k$; and by $[g_1, l_r]$ followed by $[g_1, k_p]$ we see that it is transformed into an equivalent form if $p, q$ are interchanged and $i, k, l$ are permuted cyclically. Since the permutations $(ik)$ and $(ikl)(pq)$ generate the direct product of the symmetric groups on the symbols $i, k, l$ and the symbols $p, q$, it follows that $i, k, l$ can be interchanged in any manner and $p, q$ permuted in any manner, all the forms so obtained being equivalent. Hence $p, q, r$ must all be distinct from $i$, $k$, $l$ for an irreducible form, while $p$, $q$, $r$ and $i$, $k$, $l$ are triads of different symbols. For five quadratics this is impossible. The same argument applies to the dual forms.

Next, the forms $(iju)\begin{pmatrix} i & k & l \\ p & q & x \end{pmatrix}(j)$ are all reducible. For the argument just used shows that all the forms obtained from the given one by permuting $(i, k, l)$ and $(p, q)$ among themselves are equivalent. Now it was shown in § 7 that

$$\Sigma\begin{pmatrix} & i & k & l \\ x & p & q & x \end{pmatrix} \equiv 0, \qquad (18\cdot1)$$

the summation extending over the six permutations of $i$, $k$, $l$. On polarizing with respect to $y$ and replacing $y$ by $(ju)(j)$ we obtain

$$\sum_{i,k,l}\left[(iju)\begin{pmatrix} i & k & l \\ p & q & x \end{pmatrix}(j)+(lju)\begin{pmatrix} l & k & i \\ q & p & x \end{pmatrix}(j)\right] \equiv 0. \qquad (18\cdot2)$$

By what has been said above, the twelve forms on the left of (18·2) are all equivalent. Hence each of them is reducible. A similar argument applies to the dual forms.

19. The next set of forms to be considered is $(iju)\begin{pmatrix} i & k & u \\ p & q & \end{pmatrix} j_\xi$, where $j_\xi$ is a tag not longer than the tag $(i)$. By $[g_1, k_q]$ this form is symmetric with respect to interchange of $i$ and $k$, and may be denoted by $[j_{pq}]$. Now

$$\begin{aligned} 0 &\equiv (iju)^2.(\mathbf{pq}\xi)\,k_p k_q \equiv (iju)\,k_p k_q[-i_p j_\xi u_q + i_q j_\xi u_p + i_\xi j_p u_q - i_\xi j_q u_p] \\ &= -[j_{pq}]+[j_{qp}]-[i_{pq}]+[i_{qp}]. \end{aligned}$$

Hence $$[i_{pq}]-[i_{qp}] \equiv -([j_{pq}]-[j_{qp}]). \qquad (19\cdot1)$$

The expression on the left of (19·1) is thus symmetric in $(jk)$ and skew in $(ij)$. Hence it is reducible (cf. § 4). Thus $[i_{pq}]$ is symmetric in $p$ and $q$, and we may write $[i]$ instead of $[i_{pq}]$ or $[i_{qp}]$.

Next consider the form $(iju)\begin{pmatrix} i & k \\ p & \xi \end{pmatrix}\begin{pmatrix} j & u \\ & q \end{pmatrix}$. By $[g_1, k_\xi]$ this form is seen to be symmetric in $i$ and $k$, so that it may be denoted by $[j'_{pq}]$. Now, by $[g_1, k_p]$,

$$[i] = (jiu)\begin{pmatrix} j & k & u \\ p & q & \end{pmatrix} i_\xi \equiv (jku)\begin{pmatrix} j & i \\ p & \xi \end{pmatrix}\begin{pmatrix} k & u \\ & q \end{pmatrix} + (jik)\begin{pmatrix} j & u \\ p & \end{pmatrix}\begin{pmatrix} k & u \\ & q \end{pmatrix} i_\xi. \qquad (19\cdot2)$$

The first term on the right of (19·2) is $[k'_{pq}]$. The second is changed in sign if $jk$ are interchanged and at the same time $pq$ are interchanged. Hence, as $[i]$ is symmetric in $jk$ and $pq$,

$$2[i] \equiv [k'_{pq}]+[j'_{qp}] \equiv [j'_{pq}]+[k'_{qp}]. \qquad (19\cdot3)$$

Further, $$0 \equiv (ijk)^2.(\mathbf{pq}\xi)\,u_p u_q = \Sigma\,(ijk)\,(j_p k_q - j_q k_p)\,i_\xi u_p u_q,$$

the sum extending over the three terms of a cyclic permutation of $ijk$; and by $[g_1, u_p]$

$$\begin{aligned} (ijk)\,(j_p k_q - j_q k_p)\,i_\xi u_p u_q &\equiv (jku)\left[\begin{pmatrix} j & i \\ p & \xi \end{pmatrix}\begin{pmatrix} k & u \\ & q \end{pmatrix} - \begin{pmatrix} k & i \\ p & \xi \end{pmatrix}\begin{pmatrix} j & u \\ & q \end{pmatrix}\right] \\ &\qquad - (kiu)\begin{pmatrix} k & j & u \\ p & q & \end{pmatrix} i_\xi - (jiu)\begin{pmatrix} j & k & u \\ p & q & \end{pmatrix} i_\xi \\ &= [k'_{pq}]+[j'_{pq}]-2[i]. \end{aligned}$$

Thus, by addition, and division by 2,

$$[i]+[j]+[k]\equiv[i'_{pq}]+[j'_{pq}]+[k'_{pq}]. \tag{19·4}$$

From (19·3) and (19·4) it is easily seen that

$$[i'_{pq}]\equiv[i'_{qp}]\equiv[i'],\text{ say,} \tag{19·5}$$

and that

$$[i]\equiv\tfrac{1}{2}([j']+[k']). \tag{19·6}$$

The reduction thus far applies equally to the dual forms. This is not the case with the next, and final stage, for which, accordingly, the dual forms require separate consideration.

We have

$$0\equiv(ipu)^2.(jk\,\mathbf{q}\xi)(jkp')p'_q\equiv(ipu)(jkp')p'_q[-(ijk)p_\xi u_q+i_qp_\xi(jku)+i_\xi(pjk)u_q-i_\xi(jku)p_q]$$
$$=T_1+T_2+T_3+T_4,\text{ say.}$$

Now

$$T_1=-[(ipu)(jkp)(ijk).p'_qp'_\xi u_q+(ipu)(ijk)p'_qu_q(\mathbf{p}\xi jk)]\equiv-\tfrac{1}{2}(ijk)(\mathbf{p}\xi jk)(\mathbf{p}\,ui\,\mathbf{q})u_q$$
$$\equiv\tfrac{1}{2}(ijk)[j_pk_\xi-j_\xi k_p]i_qu_pu_q=\tfrac{1}{2}[(jku)i_p+(kiu)j_p+(iju)k_p][j_pk_\xi-j_\xi k_p]i_qu_q$$
$$\equiv\tfrac{1}{2}\Big[(jku)\begin{pmatrix}j&&i&&u\\&p&&q&\end{pmatrix}k_\xi+(kju)\begin{pmatrix}k&&i&&u\\&p&&q&\end{pmatrix}j_\xi-(kiu)\begin{pmatrix}k&&j&\\&p&&\xi\end{pmatrix}\begin{pmatrix}i&&u\\&q&\end{pmatrix}-(jiu)\begin{pmatrix}j&&k&\\&p&&\xi\end{pmatrix}\begin{pmatrix}i&&u\\&q&\end{pmatrix}\Big]$$
$$\equiv\tfrac{1}{2}([k]+[j]-[i']-[i']),$$

$$T_2=[(jkp')(jku)p'_\xi.(ipu)i_qp_q+(jkp')(jku)(ipu)(\mathbf{p}\xi\mathbf{q})i_q]\equiv\tfrac{1}{2}(jku)(\mathbf{p}\xi\mathbf{q})(\mathbf{p}\,ui\,jk)i_q$$
$$\equiv\tfrac{1}{2}[j_pk_\xi u_q+j_\xi k_qu_p-j_\xi k_pu_q-j_qk_\xi u_p][(iku)j_p-(iju)k_p]i_q$$
$$\equiv\tfrac{1}{2}\Big[(kiu)\begin{pmatrix}k&&j&\\&p&&\xi\end{pmatrix}\begin{pmatrix}i&&u\\&q&\end{pmatrix}-(iku)\begin{pmatrix}i&&j&&u\\&q&&p&\end{pmatrix}k_\xi+(jiu)\begin{pmatrix}j&&k&\\&p&&\xi\end{pmatrix}\begin{pmatrix}i&&u\\&q&\end{pmatrix}-(iju)\begin{pmatrix}i&&k&&u\\&q&&p&\end{pmatrix}j_\xi\Big]$$
$$\equiv\tfrac{1}{2}([i']-[k]+[i']-[j])\equiv-T_1,$$

$$T_3\equiv-\tfrac{1}{2}(\mathbf{p}jk\,ui)(\mathbf{p}jk\,\mathbf{q})i_\xi u_q\equiv\tfrac{1}{2}[(iju)k_q+(iku)j_q]j_pk_pi_\xi u_q\equiv\tfrac{1}{2}(-[i]-[i])\equiv-[i],$$

$$T_4\equiv\tfrac{1}{2}(\mathbf{pq}\,ui)(\mathbf{pq}\,jk)(jku)i_\xi=\tfrac{1}{2}[u_pi_q-u_qi_p][j_pk_q-j_qk_p](jku)i_\xi$$
$$=\tfrac{1}{2}\Big[-(kju)\begin{pmatrix}k&&i&\\&q&&\xi\end{pmatrix}\begin{pmatrix}j&&u\\&p&\end{pmatrix}-(jku)\begin{pmatrix}j&&i&\\&p&&\xi\end{pmatrix}\begin{pmatrix}k&&u\\&q&\end{pmatrix}$$
$$-(jku)\begin{pmatrix}j&&i&\\&q&&\xi\end{pmatrix}\begin{pmatrix}k&&u\\&p&\end{pmatrix}-(kju)\begin{pmatrix}k&&i&\\&p&&\xi\end{pmatrix}\begin{pmatrix}j&&u\\&q&\end{pmatrix}\Big]$$
$$\equiv-\tfrac{1}{2}([j']+[k']+[k']+[j'])\equiv-2[i],\text{ by (19·6)}$$

and hence, adding, $[i]\equiv0$.

The duals of the forms defined in this section are

$$[i]\equiv(\mathbf{ji}x)\begin{pmatrix}&p&&q&\\j&&k&&x\end{pmatrix}\omega_i,\quad [i']\equiv(\mathbf{ji}x)\begin{pmatrix}&p&&\omega\\j&&k&\end{pmatrix}\begin{pmatrix}&q&\\i&&x\end{pmatrix},$$

and $[i]$, $[i']$ are symmetric in $p$, $q$ and in $j$, $k$ and satisfy (19·6). To reduce them consider the relation

$$0\equiv(\mathbf{jk}x)i_j\omega_ki_x.(i'pq)^2\equiv(i'pq)[i'_j(pq\,\mathbf{k}x)+i'_k(pq\,x\mathbf{j})+i'_x(pq\,\mathbf{jk})]i_j\omega_ki_x$$
$$=T_1+T_2+T_3,\text{ say.}$$

Then $$T_1 \equiv -\tfrac{1}{2}(\mathbf{ij}x)\,(\mathbf{ij}\,pq)\,(pq\,\mathbf{k}x)\,\omega_k = -\tfrac{1}{2}(\mathbf{ij}x)\,[p_i q_j - p_j q_i]\,[p_k q_x - p_x q_k]\,\omega_k$$
$$\equiv -\tfrac{1}{2}([j'] + [i'] + [i'] + [j']) = -([i'] + [j']),$$

$$T_2 = i_k \omega_k i_x . (i'pq)\, i'_j\,(pq\,x\mathbf{j}) + (i'pq)\,(pq\,x\mathbf{j})\,(\mathbf{ijk})\,\omega_k i_x$$
$$\equiv \tfrac{1}{2}(\mathbf{i}x\,pq)\,(pq\,x\mathbf{j})\,(\mathbf{ijk})\,\omega_k \equiv \tfrac{1}{2}(\mathbf{ijk})\,[p_i q_j + p_j q_i]\,p_x q_x \omega_k$$
$$\equiv \tfrac{1}{2}[(\mathbf{jk}x)\,p_i + (\mathbf{ki}x)\,p_j + (\mathbf{ij}x)\,p_k]\,[p_i q_j + p_j q_i]\,q_x \omega_k$$
$$\equiv \tfrac{1}{2}([k] - [k] + [j'] - [i']) \equiv \tfrac{1}{2}([j'] - [i']),$$

$$T_3 \equiv -\tfrac{1}{2}(\mathbf{i}x\mathbf{j})\,(\mathbf{i}x\,pq)\,(pq\,\mathbf{jk})\,\omega_k = -\tfrac{1}{2}(\mathbf{ji}x)\,[p_i q_x - p_x q_i]\,[p_j q_k - p_k q_j]\,\omega_k$$
$$\equiv -\tfrac{1}{2}(0 + [j'] + [j'] + 0) \equiv -[j'].$$

Hence, by addition, $[i'] + [j'] \equiv 0$. Hence, permuting $i, j, k$, $[i'] \equiv 0$ and so, from (19·6), $[i] \equiv 0$.

Thus all the forms reduce. It will be observed that the forms $[i']$ are also reducible, since (19·6) implies $[i'] \equiv [j] + [k] - [i]$.

20. Consider next $(iju)\begin{pmatrix} i & & k & \\ & p & & x \end{pmatrix}(j)$, where $(j)$ is a tag not longer than $(i)$. If $(j)$ is $\begin{pmatrix} j & & l & \\ & q & & x \end{pmatrix}$ the form is reducible. For we may suppose $i, j, k, l$ all distinct and $p$ different from $q$ (cf. § 18). For five or fewer quadratics, one of $p$, $q$ must coincide with one of $i, j, k, l$. We may assume, without loss of generality, that $q = k$, since $[g_1, k_x]$ shows that the form is symmetrical in $i$ and $k$. By $[g_1, k_p]$,

$$(iju)\begin{pmatrix} i & & k & \\ & p & & x \end{pmatrix}\begin{pmatrix} j & & l & \\ & k & & x \end{pmatrix} \equiv (iku)\begin{pmatrix} i & & j & & l & \\ & p & & k & & x \end{pmatrix} k_x + (ijk)\begin{pmatrix} i & & u \\ & p & \end{pmatrix}\begin{pmatrix} j & & l & \\ & k & & x \end{pmatrix} k_x.$$

The first term on the right reduces by § 18, and the second, by $[g_1, l_k]$, is equivalent to $(ljk)\begin{pmatrix} j & & i & & u \\ & k & & p & \end{pmatrix} k_x l_x$ which, by $[g_1, u_p]$ is expressible as the sum of forms shown to be reducible.

Hence $(iju)\begin{pmatrix} i & & k & \\ & p & & x \end{pmatrix}\begin{pmatrix} j & & l & \\ & q & & x \end{pmatrix} \equiv 0$, and the dual form is similarly reducible.

If $(j)$ is $\begin{pmatrix} j & & u \\ & q & \end{pmatrix}$ the form is that denoted by $[j']$ in § 19, and so this form, and its dual, reduces. Finally, if $(j)$ is $j_x$ the form belongs to four quadratics at most; it is symmetric in $i$ and $k$, and the dual form is reducible (Todd 1948). So the only irreducible sets are

[26] $(01u)\begin{pmatrix} 1 & & 2 & \\ & 0 & & x \end{pmatrix} 0_x$ Orders (2, 1). Degrees $\{31^2\}$. One form.

[27] $(12u)\begin{pmatrix} 1 & & 3 & \\ & 0 & & x \end{pmatrix} 2_x$ Orders (2, 1). Degrees $\{21^3\}$. Three forms.

21. The remaining forms $(iju)\,(i)\,(j)$ are of one or other of the types

$$(iju)\begin{pmatrix} i & & u \\ & p & \end{pmatrix}\begin{pmatrix} j & & u \\ & q & \end{pmatrix}, \quad (iju)\begin{pmatrix} i & & u \\ & p & \end{pmatrix} j_x, \quad (iju)\, i_x j_x$$

and all belong to less than five quadratics. These types are known (Turnbull 1910) and give rise to the following sets of irreducible forms:

[28] $(01u)\begin{pmatrix} 0 & & u \\ & 1 & \end{pmatrix}\begin{pmatrix} 1 & & u \\ & 0 & \end{pmatrix}$ Orders $(0, 3)$. Degrees $\{3^2\}$. One form.

[29] $(\mathbf{01}x)\begin{pmatrix} & 1 & \\ 0 & & x \end{pmatrix}\begin{pmatrix} & 0 & \\ 1 & & x \end{pmatrix}$ Orders $(3, 0)$. Degrees $\{3^2\}$. One form.

[30] $(02u)\begin{pmatrix} 0 & & u \\ & 1 & \end{pmatrix}\begin{pmatrix} 2 & & u \\ & 0 & \end{pmatrix}$ Orders $(0, 3)$. Degrees $\{321\}$. One form.

[31] $(\mathbf{01}x)\begin{pmatrix} & 2 & \\ 0 & & x \end{pmatrix}\begin{pmatrix} & 0 & \\ 1 & & x \end{pmatrix}$ Orders $(3, 0)$. Degrees $\{321\}$. One form.

[32] $(23u)\begin{pmatrix} 2 & & u \\ & 0 & \end{pmatrix}\begin{pmatrix} 3 & & u \\ & 1 & \end{pmatrix}$ Orders $(0, 3)$. Degrees $\{2^2 1^2\}$. Two forms.

[33] $(\mathbf{01}x)\begin{pmatrix} & 2 & \\ 0 & & x \end{pmatrix}\begin{pmatrix} & 3 & \\ 1 & & x \end{pmatrix}$ Orders $(3, 0)$. Degrees $\{2^2 1^2\}$. Two forms.

[34] $(01u)\begin{pmatrix} 1 & & u \\ & 0 & \end{pmatrix} 0_x$ Orders $(1, 2)$. Degrees $\{31\}$. One form.

[35] $(\mathbf{01}x)\begin{pmatrix} & 0 & \\ 1 & & x \end{pmatrix} u_0$ Orders $(2, 1)$. Degrees $\{32\}$. One form.

[36] $(12u)\begin{pmatrix} 1 & & u \\ & 0 & \end{pmatrix} 2_x$ Orders $(1, 2)$. Degrees $\{21^2\}$. Two forms.

[37] $(\mathbf{01}x)\begin{pmatrix} & 2 & \\ 0 & & x \end{pmatrix} u_1$ Orders $(2, 1)$. Degrees $\{2^2 1\}$. Two forms.

[38] $(01u)\, 0_x 1_x$ Orders $(2, 1)$. Degrees $\{1^2\}$. One form.

[39] $(\mathbf{01}x)\, u_0 u_1$ Orders $(1, 2)$. Degrees $\{2^2\}$. One form.

*The subsystem* $B_6$

22. These are the forms $(iju)\,(jku)\,Q$, where $Q$ is a chain $(i, k)$ or the product of two tags $(i)\,(k)$. In the latter case, as for the system $B_5$, the upper symbols in the tags are all distinct and the lower symbols are all distinct, or else the form reduces. Moreover, $j$ cannot occur as an upper symbol in a chain or tag, since if it did the form would contain a form $B_5$ as a factor.

If $Q$ is a chain $(i, k)$ there are three possibilities, according as the chain is

$$\text{(i)}\ \begin{pmatrix} i & & k \\ & p & \end{pmatrix}, \qquad \text{(ii)}\ \begin{pmatrix} i & & l & & k \\ & p & & q & \end{pmatrix}, \qquad \text{(iii)}\ \begin{pmatrix} i & & l & & m & & k \\ & p & & q & & r & \end{pmatrix}.$$

The third type reduces at once by writing $(ju)$ for $\xi$ and $\eta$ in the relation

$$\begin{matrix} & i & & l & & m & & k & \\ \xi & & p & & q & & r & & \eta \end{matrix} \equiv 0$$

established in §12. Its dual reduces similarly. The first type belongs to four quadratics (or less) and yields the sets (Turnbull 1910)

[40] $(13u)\,(32u)\begin{pmatrix} 1 & & 2 \\ & 0 & \end{pmatrix}$ Orders $(0, 2)$. Degrees $\{21^3\}$. Two forms.

[41] $(\mathbf{01}x)\,(\mathbf{12}x)\begin{pmatrix} & 3 & \\ 0 & & 2 \end{pmatrix}$ Orders $(2, 0)$. Degrees $\{2^3 1\}$. Two forms.

The system (ii) remains to be discussed.

23. In the form $(iju)\,(jku)\begin{pmatrix} i & l & k \\ p & q & \end{pmatrix}$, we have seen that $i, j, k, l$ are all different and that $p$, $q$ are different. Hence at least one symbol is common to the two sets $(i, j, k, l)$ and $(p, q)$. If this symbol is denoted by $i$ we have to consider the two types of form

$$(iju)\,(jku)\begin{pmatrix} i & l & k \\ p & i & \end{pmatrix} = [lk], \text{ say,} \tag{23·1}$$

$$(jiu)\,(iku)\begin{pmatrix} j & l & k \\ p & i & \end{pmatrix} = [jlk], \text{ say,} \tag{23·2}$$

each type giving rise to six forms corresponding to the permutations of $j$, $k$, $l$. We shall find it convenient to consider the auxiliary form

$$(iju)\,(klu)\begin{pmatrix} i & k \\ p & \end{pmatrix}\begin{pmatrix} j & l \\ i & \end{pmatrix} = [kl]', \text{ say.} \tag{23·3}$$

We proceed to consider the relations between these forms. First, by replacing $x$ by $(lu)$ in the relation

$$\begin{matrix} & i & j & k & \\ x & p & i & x \end{matrix} + \begin{matrix} & i & k & j & \\ x & p & i & x \end{matrix} \equiv 0$$

of § 7, we obtain

$$[jk]+[kj] \equiv 0. \tag{23·4}$$

Next,

$$0 \equiv \tfrac{1}{2}(\mathbf{ip}\,jk)^2 . (ilu)^2 \equiv \tfrac{1}{2}\begin{vmatrix} \cdot & i_p & (ijk) \\ l_i & l_p & (ljk) \\ u_i & u_p & (jku) \end{vmatrix}^2$$

$$\equiv i_p(ijk)\, l_i u_i[l_p(jku)+u_p(ljk)]$$

$$\equiv i_p l_i[(kiu)j_i+(iju)\,k_i]\,[2(jku)\,l_p+(klu)j_p+(lju)\,k_p]$$

$$\equiv 2[lj]-[jl]+2[lk]-[kl] \equiv 3[lj]+3[lk], \text{ by (23·4).}$$

Hence

$$[lj]+[lk] \equiv 0; \tag{23·5}$$

and (23·4) and (23·5) show that $[jk]$ is skew with respect to each pair of $j$, $k$, $l$. Next,

$$[jk]'+[kj]' = (ilu)\,(jku)\left[\begin{pmatrix} i & j \\ p & \end{pmatrix}\begin{pmatrix} l & k \\ i & \end{pmatrix} - \begin{pmatrix} i & k \\ p & \end{pmatrix}\begin{pmatrix} l & j \\ i & \end{pmatrix}\right] = (ilu)\,(jku)\, i_p l_i(jk\,\mathbf{pi})$$

$$\equiv (jku)\, i_p l_i(ijk)\, l_p u_i \equiv (jku)\, i_p l_i l_p[(kiu)j_i+(iju)\,k_i]$$

$$= [lj]+[lk] \equiv 0, \text{ by (23·5),}$$

and

$$0 \equiv (jku)j_ik_i . (ilu)\, i_p l_p \equiv [(iju)\,(lku)+(iku)\,(jlu)]j_ik_ii_pl_p = [lk]'-[lj]'.$$

Thus the form $[jk]'$ is skew in $j$, $k$ and symmetric in $k$ and $l$. Hence (cf. § 4)

$$[jk]' \equiv 0. \tag{23·6}$$

Next,

$$0 \equiv (ijk)\,(ijl)\, k_p l_p . u_i^2 \equiv [(kiu)j_i+(iju)\,k_i]\,[(liu)j_i+(iju)\,l_i]\, k_p l_p$$

$$\equiv [klj]+[lkj],$$

so that

$$[klj]+[lkj] \equiv 0. \tag{23·7}$$

And

$$0 \equiv [jk]' \equiv (ilu)\,(jku)\begin{pmatrix} i & j \\ p & \end{pmatrix}\begin{pmatrix} l & k \\ i & \end{pmatrix} \equiv (ilu)\left[(lku)\begin{pmatrix} j & k \\ i & \end{pmatrix}+(jkl)\begin{pmatrix} u & k \\ i & \end{pmatrix}\right]\begin{pmatrix} i & j \\ p & \end{pmatrix}$$

$$\equiv [jk]+(ilu)\,(jkl)\begin{pmatrix} u & k \\ i & \end{pmatrix}\begin{pmatrix} i & j \\ p & \end{pmatrix}.$$

Hence $[jk] \equiv -(ilu)\,(jkl)\begin{pmatrix} u & & k \\ & i & \end{pmatrix}\begin{pmatrix} i & & j \\ & p & \end{pmatrix} \equiv -(ilu)\begin{pmatrix} u & & k \\ & i & \end{pmatrix}\left[(jil)\begin{pmatrix} k & & j \\ & p & \end{pmatrix} + (jki)\begin{pmatrix} l & & j \\ & p & \end{pmatrix}\right]$

$$\equiv -(ilu)\,(jiu)\begin{pmatrix} j & & k & & l \\ & p & & i & \end{pmatrix} - (ilu)\,(kiu)\begin{pmatrix} l & & j & & k \\ & p & & i & \end{pmatrix}$$

$$\equiv -[jkl] - [ljk]$$

so that
$$[jk] \equiv -[jkl] - [ljk]. \tag{23·8}$$

Interchanging $j$ and $k$ in (23·8), and using (23·4) and (23·7), we obtain
$$[ljk] + [lkj] \equiv 0. \tag{23·9}$$

From (23·7) and (23·9) it follows that $[jkl]$ is skew in $j, k, l$, so that (23·8) may be replaced by
$$[jk] \equiv -2[jkl]. \tag{23·10}$$

The relations so far obtained, which hold equally for the dual forms, show that the twelve forms obtained by permuting $j, k, l$ in (23·1) and (23·2) may be expressed in terms of any one of them. Actually, these forms are all reducible. To see this, replace $s, r, i, j, p, q$ in (8·12) by $i, j, k, l, i, p$ respectively. Then (8·12) becomes

$$(ijk)\,(ijl)\,[k_i l_p + 7k_p l_i]\,u_i u_p + (ijk)\,(iju)\,[4k_i u_p - 2k_p u_i]\,l_i l_p - 2(ijl)\,(iju)\,[l_i u_p + l_p u_i]\,k_i k_p \equiv 0.$$

The first term on the left, by $[g_1, u_i]$ and $[g_2, u_p]$, is equivalent to

$$(kiu)\,j_i[(jlu)\,i_p + (liu)\,j_p]\,k_i l_p = [lj]' - [ljk] \equiv -[jkl],$$

since $[lj]' \equiv 0$ and $[jkl]$ is skew in $j, k, l$. By interchanging $k$ and $l$ we see that the second term is equivalent to
$$-7[jlk] \equiv 7[jkl].$$

The third and fourth terms, by $[g_1, u_p]$ and $[g_1, u_i]$ respectively, give

$$4[(kiu)\,j_p + (jku)\,i_p]\,(iju)\,k_i l_i l_p = 4([jlk] + [lk]) \equiv -4[jlk] \equiv 4[jkl],$$

and
$$-2(kiu)\,j_i(iju)\,k_p l_i l_p = -2[klj] \equiv -2[jkl].$$

By interchange of $k$ and $l$ the last two terms give, respectively,

$$-2[jlk] \equiv 2[jkl] \quad \text{and} \quad -2[jlk] \equiv 2[jkl].$$

Hence, by addition, $12[jkl] \equiv 0$. Hence $[jkl] \equiv 0$ and the forms are all reducible.

For the dual forms, using the corresponding notation

$$[jkl] \equiv (\mathbf{ji}x)\,(\mathbf{il}x)\begin{pmatrix} & p & & i & \\ j & & k & & l \end{pmatrix}, \quad [kl] \equiv (\mathbf{ij}x)\,(\mathbf{jl}x)\begin{pmatrix} & p & & i & \\ i & & k & & l \end{pmatrix}, \quad [kl]' \equiv (\mathbf{ij}x)\,(\mathbf{kl}x)\begin{pmatrix} & p & \\ i & & k \end{pmatrix}\begin{pmatrix} & i & \\ j & & l \end{pmatrix},$$

the equivalences (23·4) to (23·10) are still valid. Now

$$0 \equiv (\mathbf{il}x)\,j_i j_l \,.\, (j'ip)\,j'_k p_k i_x \equiv i_l(p_x j'_i - p_i j'_x)\,j_i j_l j'_k p_k i_x = T_1 + T_2, \text{ say.}$$

And

$$T_1 \equiv -\tfrac{1}{2}(\mathbf{jil})\,(\mathbf{jik})\,p_x p_k i_l i_x \equiv -\tfrac{1}{2}(\mathbf{il}x)\,(\mathbf{jik})\,p_x p_k i_l i_j \equiv -\tfrac{1}{2}(\mathbf{il}x)\,[(\mathbf{ik}x)\,p_j + (\mathbf{kj}x)\,p_i]\,p_k i_l i_j$$
$$= \tfrac{1}{2}[kjl] - \tfrac{1}{2}[kj]' \equiv -\tfrac{1}{2}[jkl],$$

$$T_2 \equiv -i_l i_x j_l j_x \,.\, p_i p_k j'_i j'_k - i_l i_x p_i p_k j_l j'_k(\mathbf{ji}x) \equiv -\tfrac{1}{2}(\mathbf{ji}x)\,(\mathbf{jlk})\,p_i p_k i_l i_x$$
$$\equiv -\tfrac{1}{2}(\mathbf{ji}x)\,[(\mathbf{lk}x)\,i_j + (\mathbf{jl}x)\,i_k]\,p_i p_k i_l = -\tfrac{1}{2}[kl]' + \tfrac{1}{2}[kl] \equiv -[jkl].$$

Hence $[jkl] \equiv 0$ and the dual forms reduce also.

24. Next consider the forms $(iju)\,(jku)\,(i)\,(k)$. We may write this in the form $(iju)\,(jku)\,i_\xi k_\eta$. By $[g_1, k_\eta]$ we see that the form is skew in $j$ and $k$, and by $[g_2, i_\xi]$ we see that it is skew in $i$ and $j$. Hence it is skew symmetric in any pair of $i, j, k$.

Suppose, now, that the tag $(k)$ is not longer than $(i)$, as we may without loss of generality. Since neither $j$ nor $k$ can occur as upper symbols in $(i)$ the tag $(i)$ has one of the forms

$$\text{(i)}\quad \begin{pmatrix} i & & l & \\ & p & & \zeta \end{pmatrix},\qquad \text{(ii)}\quad \begin{pmatrix} i & & u \\ & p & \end{pmatrix},\qquad \text{(iii)}\quad \begin{pmatrix} i & \\ & x \end{pmatrix},$$

where $\zeta$ is $(x)$ or some cogredient combination. But the form $(iju)\,(jku)\begin{pmatrix} i & & l & \\ & p & & \zeta \end{pmatrix} k_\eta$ is symmetric in $i$ and $l$, as may be seen by applying $[g_1, l_\zeta]$. Since it is skew in $i$ and $j$ it must therefore be reducible. The same argument applies to the dual forms. Moreover, if $(i)$ is $i_x$, $(k)$, which is not longer than $(i)$, must be $k_x$, and the form $(iju)\,(jku)\,i_x k_x$ is symmetric as well as skew symmetric in $i$, $k$ and so reduces. Hence $(i)$ must be $\begin{pmatrix} i & & u \\ & p & \end{pmatrix}$, whence $(k)$ is $\begin{pmatrix} k & & u \\ & q & \end{pmatrix}$ or $\begin{pmatrix} k & \\ & x \end{pmatrix}$. The latter form belongs to less than five quadratics. It gives rise to the irreducible set

[42] $(12u)\,(23u)\begin{pmatrix} 1 & & u \\ & 0 & \end{pmatrix} 3_x$ Orders $(1, 3)$. Degrees $\{21^3\}$. One form,

the dual form being reducible (Todd 1948).

There remains for consideration the form $(iju)\,(jku)\begin{pmatrix} i & & u \\ & p & \end{pmatrix}\begin{pmatrix} k & & u \\ & q & \end{pmatrix}$ and its dual. Both these forms are skew in $i, j, k$. To reduce the given form, we have

$$\begin{aligned} 0 &\equiv (jku)\,(kpu)\,(jpu)\,.\,i_q p'_q (ip'u) \equiv (kpu)\,(jpu)\,[(iku)\,(jp'u) - (iju)\,(kp'u)]\,i_q p'_q \\ &\equiv -\tfrac{1}{2}(\mathbf{p}\,uj\,uk)\,(\mathbf{p}\,uj\,\mathbf{q})\,(iku)\,i_q + \tfrac{1}{2}(\mathbf{p}\,uk\,uj)\,(\mathbf{p}\,uk\,q)\,(iju)\,i_q \\ &\equiv \tfrac{1}{2}(kju)\,u_q u_p j_p (iku)\,i_q - \tfrac{1}{2}(jku)\,u_q u_p k_p (iju)\,i_q \\ &= \tfrac{1}{2}(iku)\,(kju)\begin{pmatrix} i & & u \\ & q & \end{pmatrix}\begin{pmatrix} j & & u \\ & p & \end{pmatrix} - \tfrac{1}{2}(iju)\,(jku)\begin{pmatrix} i & & u \\ & q & \end{pmatrix}\begin{pmatrix} k & & u \\ & p & \end{pmatrix} \\ &\equiv (iku)\,(kju)\begin{pmatrix} i & & u \\ & q & \end{pmatrix}\begin{pmatrix} j & & u \\ & p & \end{pmatrix} \text{ by skew symmetry in } j \text{ and } k. \end{aligned}$$

The dual form reduces even more simply, since

$$0 \equiv i_x i_j p_x p_j\,.\,i'_x i'_k q_x q_k \equiv -\tfrac{1}{2}(\mathbf{ji}x)\,(\mathbf{ki}x)\,p_x p_j q_x q_k = \tfrac{1}{2}(\mathbf{ji}x)\,(\mathbf{ik}x)\begin{pmatrix} & p & \\ j & & x \end{pmatrix}\begin{pmatrix} & q & \\ k & & x \end{pmatrix}.$$

*The subsystems $B_7$ and $B_8$*

25. The forms $B_7$ are $(iju)\,(jku)\,(klu)\,Q$, where $Q$ is a chain $(i, l)$ or the product of two tags $(i)\,(l)$. The latter forms are reducible, since by applying in succession $[g_1, k \text{ of } g_3]$, $[g_1, l_\eta]$, $[g_1, j \text{ of } g_2]$, $[g_3, k_\eta]$ to the form $(iju)\,(jku)\,(klu)\,i_\xi l_\eta$ the form appears as equivalent to its negative (cf. Turnbull 1910, p. 104, whose argument is slightly different); the same reasoning applies to the dual forms. Hence $Q$ must be a chain, and since $j$ and $k$ cannot occur as upper links in an irreducible form $Q$ is either $\begin{pmatrix} i & & l \\ & p & \end{pmatrix}$ or $\begin{pmatrix} i & & m & & l \\ & p & & q & \end{pmatrix}$.

Consider then the form $(iju)\,(jku)\,(klu)\begin{pmatrix} i & & l \\ & p & \end{pmatrix} = [ijkl]$, say. (25·1)

Clearly $[ijkl] = -[lkji]$, and by $[g_1, k \text{ of } g_3]$, $[ijkl] \equiv -[ikjl]$. Hence the form is skew in $j$, $k$ and symmetric in $i$, $l$. If $a$, $b$, $c$, $d$ denote the symbols $i, j, k, l$ taken in some fixed order, we may thus confine our attention to these forms (25·1) in which $(ijkl)$ is a positive permutation of $(abcd)$, and denote such a form by the simpler symbol $[il]$. Since $[ijkl] = -[lkji]$ we then have $[il]+[li] = 0$.

By $[g_1, l_p]$, $$[ijkl] \equiv [ilkj]+(ijl)\,(jku)\,(klu)\begin{pmatrix} i & & u \\ & p & \end{pmatrix}.$$

By applying $[g_2, i_p]$, $[g_1, u_p]$ in succession to the second term it is easily seen that

$$\begin{aligned}[ijkl] &\equiv [ilkj]-[iklj]-[jikl]+[ijlk]+[jilk]\\ &\equiv -[iklj]+[jlki]+[jkil]-[iljk]+[jilk]\end{aligned}$$

by the properties of symmetry and antisymmetry just established. Hence

$$\begin{aligned}[ij]+[ik]+[il] &\equiv [ji]+[jk]+[jl]\\ &\equiv [ki]+[kj]+[kl] \equiv [li]+[lj]+[lk]\\ &\equiv 0, \end{aligned} \tag{25·2}$$

since $[ij]+[ji] \equiv 0$. Thus all the forms are expressible in terms of $[ij]$, $[ik]$, $[jk]$. So three irreducible forms arise if $p$ is distinct from $i, j, k, l$; two of these are reducible if $p$ coincides with one of these four symbols (and the form then belongs to four quadratics). The dual form reduces since (using an evident notation for the dual forms)

$$\begin{aligned} 0 &\equiv (\mathbf{jk}x)\,i_j i_k . i'_x i'_l p_x p_l \equiv i_j i_k i'_x p_x p_l[(\mathbf{lk}x)\,i'_j + (\mathbf{jl}x)\,i'_k]\\ &\equiv -\tfrac{1}{2}[(\mathbf{ijk})\,(\mathbf{ij}x)\,(\mathbf{lk}x)\,p_x p_l + (\mathbf{ikj})\,(\mathbf{ik}x)\,(\mathbf{jl}x)\,p_x p_l]\\ &\equiv -\tfrac{1}{2}(\mathbf{ij}x)\,(\mathbf{lk}x)\,[(\mathbf{jk}x)\,p_i + (\mathbf{ki}x)\,p_j]\,p_l - \tfrac{1}{2}(\mathbf{ik}x)\,(\mathbf{jl}x)\,[(\mathbf{kj}x)\,p_i + (\mathbf{ji}x)\,p_k]\,p_l\\ &= \tfrac{1}{2}([ijkl]+[jikl]-[ikjl]-[kijl]) \equiv \tfrac{1}{2}([il]-[jl]+[il]-[kl]) \equiv \tfrac{3}{2}[il],\end{aligned}$$

as $[il]+[jl]+[kl] \equiv -[li]-[lj]-[lk] \equiv 0$, by (25·2).

Thus we obtain the irreducible sets

[43] $(10u)\,(03u)\,(32u)\begin{pmatrix} 1 & & 2 \\ & 0 & \end{pmatrix}$ Orders $(0, 3)$. Degrees $\{31^3\}$. One form.

[44] $(12u)\,(23u)\,(34u)\begin{pmatrix} 1 & & 4 \\ & 0 & \end{pmatrix}$ Orders $(0, 3)$. Degrees $\{21^4\}$. Three forms.

26. Consider next the form $(iju)\,(jku)\,(klu)\begin{pmatrix} i & & m & & l \\ & p & & q & \end{pmatrix}$. Here $i, j, k, l, m$ is a permutation of 0, 1, 2, 3, 4; and we may suppose without loss of generality that $p = 0$, $q = 1$. By $[g_1, k \text{ of } g_3]$ the form is skew in $j$ and $k$. Hence we may confine ourselves to these forms for which $(ijklm)$ is a positive permutation of (01234), and denote such a form by $[ilm]$. With this notation it should be noticed that the interchange of $i$ and $j$ in the symbolic expression of the form $[ilm]$ produces, not $[jlm]$ but $-[jlm]$, since if $ijklm$ is an even permutation $jiklm$ is odd; to get the even permutation for the corresponding form we must interchange $i$ and $k$, which changes the sign of the form.

By $[g_3, m_p]$ it is easily seen that

$$[ilm]+[iml] \equiv (iju)\,(jku)\,(klm)\begin{pmatrix} i & & u \\ & p & \end{pmatrix}\begin{pmatrix} m & & l \\ & q & \end{pmatrix}.$$

By $[g_2, i_p]$ the expression on the right is skew in $i$ and $j$. Hence

$$\begin{aligned}[ilm]+[iml] &\equiv [jlm]+[jml] \\ &\equiv [klm]+[kml],\end{aligned} \tag{26·1}$$

by symmetry. By interchanging the parts played by $p$ and $q$ we have, in the same way,

$$[ijm]+[mji] \equiv [ikm]+[mki] \equiv [ilm]+[mli]. \tag{26·2}$$

Next, by $[g_1, l_q]$, $\quad [ilm]+[ijm] \equiv (ijl)\,(jku)\,(klu)\begin{pmatrix} i & m & u \\ p & & q \end{pmatrix}.$

By $[g_2, i_p]$ followed by $[g_1, u_q]$ applied to the right-hand expression we find after a little reduction that

$$[ilm]+[ijm]+[kjm] \equiv [jlm]+[jim]+[kim]. \tag{26·3}$$

Now, with $p=0$ and $q=1$, $[ilm]$ is reducible if either $i=0$, $m=0$ or $1$, $l=1$. Let $a$, $b$, $c$ denote 2, 3, 4 in some order. Putting successively $i=0, j=1$ and $i=1, l=0$ in (26·1), and $j=0$, $l=1$ and $l=0$, $i=1$ in (26·2), we obtain

$$\begin{aligned}&[1bc]+[1cb] \equiv [abc]+[acb] \equiv 0, \quad [10a] \equiv [b0a] \equiv [c0a], \\ &[a0b]+[b0a] \equiv [acb]+[bca] \equiv 0, \quad [10a] \equiv [1ba] \equiv [1ca].\end{aligned}$$

These relations imply that $[10a] \equiv 0$, $[1bc] \equiv 0$, $[a0b] \equiv 0$, and that $[abc]$ is skew in 2, 3, 4. Putting $i=1$, $l=0$ in (26·3) it follows that $[abc] \equiv 0$. Thus all these forms are reducible. The same reasoning applies to the dual forms.

27. The forms $(iju)\,(jku)\,(klu)\,(lmu)\,Q$ of the system $B_8$ are all reducible (Turnbull 1910); the proof is analogous to that by which the system $B_4$ was reduced in § 15, with the sole difference that $Q$ replaces $(miu)$ as multiplier. The same reasoning applies to the dual forms.

*The subsystems $B_9$ and $B_{10}$*

28. The forms of the systems $B_9$, $B_{10}$ are reducible, respectively, to forms of the systems $B_6$ and $B_7$ already considered. Consider, first, the forms $B_9$, of type $(iju)\,(klu)\,Q$. The product $Q$ is composed of chains and tags, and contains each of the symbols $i, j, k, l$ once. If $Q$ contains a chain $(i, j)$ or a pair of tags $(i)\,(j)$ it is clearly reducible. If it contains a pair of tags $(i)\,(k)$ it reduces to forms of the system $B_6$ by $[g_1, k \text{ of } (k)]$. Whence $Q$, if irreducible, must be the product of two chains of the form $(i, k)\,(j, l)$. Again, the forms $B_{10}$ are of type $(iju)\,(klu)\,(lmu)\,Q$, where $Q$ contains $i, j, k, m$ once. If $Q$ contains a factor of the forms $(i, j)$, $(i)\,(j)$; $(k, m)$; $(k)\,(m)$ it is clearly reducible. If it contains $(i)\,(k)$ it reduces to forms of the system $B_7$ by $[g_1, k \text{ of } (k)]$. Hence $Q$, if irreducible, is of the form $(i, k)\,(j, m)$. Since no symbol which occurs in a bracket can occur as an upper symbol in a chain without the form reducing, the only possibilities to be considered are

$$\text{(i)} \quad (iju)\,(klu)\begin{pmatrix} i & k \\ p & \end{pmatrix}\begin{pmatrix} j & l \\ & q \end{pmatrix},$$

$$\text{(ii)} \quad (iju)\,(klu)\begin{pmatrix} i & m & k \\ p & & q \end{pmatrix}\begin{pmatrix} j & l \\ & r \end{pmatrix},$$

$$\text{(iii)} \quad (iju)\,(klu)\,(lmu)\begin{pmatrix} i & k \\ p & \end{pmatrix}\begin{pmatrix} j & m \\ & q \end{pmatrix}.$$

In considering these forms we shall use the notation $f \to g$ to indicate that $f-g$ is expressible in terms of forms of the systems $B_6$ or $B_7$ (as the case may be). The first two forms are reduced by Turnbull, but his reduction of (i), based on his 'perpetuant reduction' is not very easy to follow. As the reductions are all quite short we give them here. They apply equally to the dual forms.

29. By $[g_2, j_q]$, $[g_2, i_p]$, $[g_2, u_q]$ in order, we see that

$$(iju)\,(klu)\begin{pmatrix} i & & k \\ & p & \end{pmatrix}\begin{pmatrix} j & & l \\ & q & \end{pmatrix} \to (iju)\,(klu)\begin{pmatrix} j & & k \\ & p & \end{pmatrix}\begin{pmatrix} i & & l \\ & q & \end{pmatrix}.$$

Similarly
$$(iju)\,(klu)\begin{pmatrix} i & & k \\ & p & \end{pmatrix}\begin{pmatrix} j & & l \\ & q & \end{pmatrix} \to (iju)\,(klu)\begin{pmatrix} i & & l \\ & p & \end{pmatrix}\begin{pmatrix} j & & k \\ & q & \end{pmatrix}.$$

Hence the form $\to 0$ if either of $p, q$ is the same as $i, j, k, l$. But, if this is not the case, $p = q = m$ and the form contains the factor $(iju)\begin{pmatrix} i & & j \\ & m & \end{pmatrix}$. Hence all the forms (i) can be expressed in terms of forms of the system $B_6$.

Next, consider the form (ii). By $[g_1, l$ of $g_2]$ this is symmetric in $l$ and $j$. By $[g_1, l_r]$, $[g_1, k_q]$, $[g_1, u_r]$ we see that, to within forms of the system $B_6$, it is skew in $l$ and $k$. Hence (cf. § 4) the form is expressible in terms of forms $B_6$ and reducible forms.

For the form (iii) we may suppose $p \neq i, k$. If $p = l$ the successive transformations $[g_3, k_l]$, $[g_2, i_l]$, $[g_3, u_l]$ give a reduction to forms $B_7$. If $p = m$ a similar reduction is obtained by $[g_1, k_m]$, $[g_1, m_q]$, $[g_1, u_m]$. By $[g_1$, $m$ of $g_3]$ we see that the form is symmetrical in $j$ and $m$. Hence it equally reduces to forms $B_7$ if $p = j$. These cases exhaust the possibilities for five quadratics, since $i, j, k, l, m$ are distinct.

*Subdivision of the system of forms* C

30. The forms of the system C fall into pairs of dual forms, $(ijk)\,Q$ and $(\mathbf{ijk})\,Q'$, where $Q$ and $Q'$ do not contain brackets of the respective forms $(ijk)$ and $(\mathbf{ijk})$. We consider the first type as representative of a pair of dual forms, and classify these according to the nature of the product $Q$. If this product contains a factor $u_p$ the form reduces, by $[g_1, u_p]$, to forms of the system B. Hence $Q$ is composed of factors of type $(iju)$, $i_p$, $i_x$ only.

The product $Q$ must contain $i, j, k$ an odd number of times, and any other symbol of the same type an even number of times. If $\{ab\}$ is used as a symbol to denote one of the expressions $(abu)$, $a_p b_p$, $a_x b_x$ it follows that (to within a permutation of $i, j, k$) the form $(ijk)\,Q$ can be written as

$$(ijk)\,.\{ia\}\{ab\}\ldots\{cd\}\{dj\}\,.\{ke\}\ldots\{fg\}g_x,$$

where $\{i,j\} \equiv \{ia\}\{ab\}\ldots\{cd\}\{dj\}$ is a product in which any two consecutive factors contain a common symbol, and may be called a generalized chain, and $\{k,x\} \equiv \{ke\}\ldots\{fg\}g_x$ (a generalized tag) is similar except for the final factor $g_x$. Any form which contains a set of terms $\{ab\}$ which can be arranged to form a closed chain $\{i,i\}$ is evidently reducible; in forming such a closed chain $\{ab\}$ and $\{ba\}$ are to be regarded as equivalent, as they differ at most by a factor $-1$.

Neglecting types which are reducible from the above considerations, and types deducible from those listed by permutation of symbols, the following subsystems arise. Those to be discussed in detail are numbered $C_1, \ldots, C_8$ and the remainder are reducible trivially for the reasons stated (cf. Turnbull 1910):

$C_1$ $(ijk)\{ij\}k_x$.

$C_2$ $(ijk)\{ij\}\{ki\}i_x$.

$(ijk)\{ij\}\{ki\}\{il\}l_x$. }
$(ijk)\{ij\}\{ki\}\{il\}\{lm\}m_x$. } Reducible by $[g_1, l \text{ of } \{il\}]$.

$(ijk)\{ij\}\{kl\}l_x$. Reducible to $C_6$ by $[g_1, l \text{ of } \{kl\}]$.

$C_3$ $(ijk)\{ij\}\{kl\}\{li\}i_x$.

$(ijk)\{ij\}\{kl\}\{li\}\{im\}m_x$. Reducible by $[g_1, m \text{ of } \{im\}]$.

$C_4$ $(ijk)\{ij\}\{kl\}\{lm\}m_x$.

$C_5$ $(ijk)\{ij\}\{kl\}\{lm\}\{mi\}i_x$.

$C_6$ $(ijk)\{il\}\{lj\}k_x$.

$(ijk)\{il\}\{lj\}\{kl\}l_x$. Reducible to $C_3$ by $[g_1, l \text{ of } \{kl\}]$.

$(ijk)\{il\}\{lj\}\{kl\}\{lm\}m_x$. Reducible by $[g_1, m \text{ of } \{lm\}]$.

$C_7$ $(ijk)\{il\}\{lj\}\{km\}m_x$.

$(ijk)\{il\}\{lj\}\{km\}\{ml\}l_x$. Reducible to preceding types by $[g_1, l_x]$.

$(ijk)\{il\}\{lj\}\{km\}\{mi\}i_x$. Reducible to preceding types by $[g_1, m \text{ of } \{km\}]$.

$C_8$ $(ijk)\{il\}\{lm\}\{mj\}k_x$.

These forms will next be considered in order.

*The subsystems $C_1$, $C_2$*

31. The forms of $C_1$ all belong to fewer than five quadratics, and it is sufficient to give the irreducible sets. These are

| | | | | |
|---|---|---|---|---|
| [45] | $(012)(01u)2_x$ | Orders (1, 1). | Degrees $\{1^3\}$. | Two forms. |
| [46] | $(\mathbf{012})(\mathbf{01}x)u_2$ | Orders (1, 1). | Degrees $\{2^3\}$. | Two forms. |
| [47] | $(012)\begin{pmatrix}1 & & 2\\ & 0 & \end{pmatrix}0_x$ | Orders (1, 0). | Degrees $\{31^2\}$. | One form. |
| [48] | $(\mathbf{012})\begin{pmatrix} & 0 & \\ 1 & & 2\end{pmatrix}u_0$ | Orders (0, 1). | Degrees $\{32^2\}$. | One form. |
| [49] | $(123)\begin{pmatrix}1 & & 2\\ & 0 & \end{pmatrix}3_x$ | Orders (1, 0). | Degrees $\{21^3\}$. | Three forms. |
| [50] | $(\mathbf{012})\begin{pmatrix} & 3 & \\ 0 & & 1\end{pmatrix}u_2$ | Orders (0, 1). | Degrees $\{2^31\}$. | Three forms. |
| [51] | $(012)0_x1_x2_x$ | Orders (3, 0). | Degrees $\{1^3\}$. | One form. |
| [52] | $(\mathbf{012})u_0u_1u_2$ | Orders (0, 3). | Degrees $\{2^3\}$. | One form. |

32. The forms $C_2$ are $(ijk)\{ij\}\{ki\}i_x$. This is clearly reducible if either factor $\{ab\}$ is of the form $a_x b_x$. Hence the forms to be considered are of the types

$$\text{(i)}\quad (ijk)(iju)(kiu)\,i_x,\qquad \text{(ii)}\quad (ijk)(iju)\begin{pmatrix} i & & k \\ & p & \end{pmatrix} i_x,\qquad \text{(iii)}\quad (ijk)\begin{pmatrix} j & & i & & k \\ & p & & q & \end{pmatrix} i_x.$$

The first two belong to less than five quadratics. They give the irreducible sets

[53] $(012)(01u)(20u)\,0_x$ Orders $(1,2)$. Degrees $\{21^2\}$. One form.

[54] $(023)(02u)\begin{pmatrix} 3 & & 0 & \\ & 1 & & x \end{pmatrix}$ Orders $(1,1)$. Degrees $\{2^2 1^2\}$. Two forms.

The duals of these forms are reducible (van der Waerden 1923; Todd 1948).

Consider now the forms (iii). When written in the usual notation the symbol $i$ occurs four times in the expression. By replacing two of the symbols $i$ by $i'$ we get a system of forms, the difference between any two of which can be expressed as a form in which $i$ and $i'$ have been convolved into a symbol $\mathbf{i}$. Such a form is seen to belong to one of the sets [21], [23] or [25], and all reductions obtained in the present section will be understood to be reductions *modulo* forms of these sets.

Since $$0 \equiv (ijk)^2 . (\mathbf{pq}x)\, i_p i_q \equiv (ijk)\, i_p i_q i_x [j_p k_q - j_q k_p],$$

we see that the form (iii) is symmetric for the interchange of $p$ and $q$. Hence the form is reducible to forms [21] and [23] unless $p, q, i, j, k$ are all different. Since the expression (iii) clearly changes sign when $j$, $k$ and $p$, $q$ are interchanged, there is essentially one form (iii) for each determination of $i, j, k, p, q$; this form being symmetric in $p$, $q$ and skew in $j$, $k$. But other forms of the same degrees arise if $i$ is interchanged with $p$ or $q$, and the relations between these must be examined. We write

$$[a] \equiv (ajk)\begin{pmatrix} j & & a & & k \\ & b & & c & \end{pmatrix} a_x \equiv (ajk)\begin{pmatrix} j & & a & & k \\ & c & & b & \end{pmatrix} a_x,$$

where $abc$ is a permutation of $ipq$. Each of the forms $[a]$, $[b]$, $[c]$ is skew in $j$, $k$. We prove that the three forms are equivalent *modulo* forms [25].

We have, in fact,

$$[c] \equiv (cjk)\begin{pmatrix} j & & c & & k \\ & a & & b & \end{pmatrix} c_x \equiv (cjk)\begin{pmatrix} c' & & k \\ & b & \end{pmatrix}(c'aa')(jaa')\,c_x$$

$$\equiv (cjk)(c'aa')\,c_x\left[\begin{pmatrix} a & & k \\ & b & \end{pmatrix}(jc'a') + \begin{pmatrix} a' & & k \\ & b & \end{pmatrix}(jac')\right] \equiv 2(cjk)(c'aa')\,c_x\begin{pmatrix} a' & & k \\ & b & \end{pmatrix}(jac')$$

$$\equiv 2[(a'jk)(c'ac) + (cja')(c'ak)]\,c_x\begin{pmatrix} a' & & k \\ & b & \end{pmatrix}(jac'),$$

the last by $[g_1, a'$ of $g_2]$. The first term on the right is

$$(a'jk)\,a_c\,(\mathbf{c}x\ ja)\begin{pmatrix} a' & & k \\ & b & \end{pmatrix} \equiv (a'jk)\,a_c j_c a_x\begin{pmatrix} a' & & k \\ & b & \end{pmatrix} \equiv (ajk)\,a'_c j_c a_x\begin{pmatrix} a' & & k \\ & b & \end{pmatrix} = [a]$$

*modulo* forms [25]. Hence

$$[c]-[a] \equiv 2(cja')(c'ak)(c'ja)\begin{pmatrix} a' & & k \\ & b & \end{pmatrix} c_x$$

$$\equiv -2(cka')(c'aj)(c'ka)\begin{pmatrix} a' & & j \\ & b & \end{pmatrix} c_x \quad \text{by skew symmetry in } j \text{ and } k,$$

$$\equiv (c'ja)(c'ka)\,[(cka')\,j_b - (cja')\,k_b]\,a'_b c_x$$

$$\equiv (c'ja)(c'ka)(jka')\,c_b a'_b c_x. \qquad (32{\cdot}1)$$

But the expression on the left is skew in $c$ and $a$. Hence interchanging $c, c'$ with $a', a$ in (32·1),

$$[a]-[c] \equiv (ajc')\,(akc')\,(jkc)\,a'_b c_b a'_x. \tag{32·2}$$

Subtracting (32·2) from (32·1) and halving,

$$[c]-[a] \equiv \tfrac{1}{2}(c'ja)\,(c'ka)\,c_b a'_b[(jka')\,c_x-(jkc)\,a'_x]$$
$$\equiv \tfrac{1}{2}(c'ja)\,(c'ka)\,c_b a'_b[(cka')\,j_x+(jca')\,k_x] = T_1+T_2, \text{ say.}$$

But, by $[g_3, j \text{ of } g_1]$,

$$T_1 \equiv \tfrac{1}{2}(c'ka)\,c_b a'_b j_x[(c'ca)\,(jka')+(c'a'a)\,(ckj)]$$
$$\equiv -\tfrac{1}{4}a_c(\mathbf{cb}\;ka)\,a'_b j_x(jka')-\tfrac{1}{4}c'_a(\mathbf{a}\;c'k\;\mathbf{b})\,c_b j_x(ckj)$$
$$\equiv -\tfrac{1}{4}a_c a_b k_c a'_b j_x(jka')-\tfrac{1}{4}c'_a c'_b k_a c_b j_x(ckj)$$
$$\equiv \tfrac{1}{8}(\mathbf{abc})\,(\mathbf{ab}\;jk)\,k_c j_x+\tfrac{1}{8}(\mathbf{cba})\,(\mathbf{cb}\;kj)\,k_a j_x$$
$$\equiv 0 \quad \textit{modulo} \text{ forms } [25], \text{ by } [g_1, j_x].$$

Similarly (by interchanging $j$ and $k$) $T_2 \equiv 0$. Hence $[c] \equiv [a] \equiv [b]$ (by symmetry) and there is one irreducible set

[55] $(234)\begin{pmatrix}3 & & 2 & & 4\\ & 0 & & 1 & \end{pmatrix}2_x$ Orders (1, 0). Degrees $\{2^3 1^2\}$. One form.

The dual of the form just considered is reducible, since

$$(\mathbf{ijk})\begin{pmatrix} & p & & q & \\ j & & i & & k\end{pmatrix}u_i \equiv 2j_k j'_i(pjj')\begin{pmatrix}p & & q & \\ & i & & k\end{pmatrix}u_i \equiv 2j_k j'_i j_i(puj')\begin{pmatrix}p & & q & \\ & i & & k\end{pmatrix} \equiv 0.$$

*The subsystems $C_3$ and $C_4$*

33. The forms $(ijk)\{ij\}\{kl\}\{li\}i_x$ are all reducible. This is clear if any factor $\{ab\}$ is $a_x b_x$. If $\{kl\}$ is $(klu)$ the form is $(ijk)\,(klu)\,\{ij\}\{li\}\,i_x$ which reduces by $[g_2, i_x]$ followed by $[g_1, l \text{ of } \{li\}]$. If $\{ij\}$ is $(iju)$ the form is $(ijk)\,(iju)\,\{kl\}\{li\}\,i_x$, where $\{kl\}$ may be supposed to be of the form $k_p l_p$ by what has preceded. By $[g_1, l \text{ of } \{li\}]$ the form is symmetric in $k$ and $l$, and by $[g_2, k \text{ of } \{kl\}]$ it is skew in $k$ and $j$. Hence it is reducible. If $\{li\}$ is $(liu)$ the form is $(ijk)\,(liu)\,\{ij\}\{kl\}\,i_x$ and reduces to the type just discussed by $[g_2, j \text{ of } \{ij\}]$. Hence the only case remaining is that in which each factor $\{ab\}$ is of the form $a_p b_p$. The form is then $(ijk)\begin{pmatrix}k & & l & & i & & j\\ & p & & q & & r & \end{pmatrix}i_x$, which is seen to be reducible on polarizing the reducible form $(jku)\begin{pmatrix}k & & l & & i & & j\\ & p & & q & & r & \end{pmatrix}$ of § 17 with respect to $i_x$. The same argument applies to the dual forms. Hence the subsystem $C_3$ yields no irreducible forms.

34. The forms $C_4$ are of type $(ijk)\{ij\}\{kl\}\{lm\}m_x$. This form clearly reduces if $\{kl\}$ or $\{lm\}$ are of the form $a_x b_x$. If $\{ij\}$ is $i_x j_x$ the following possibilities arise:

(i) $(ijk)\,i_x j_x(klu)\,(lmu)\,m_x$. (ii) $(ijk)\,i_x j_x(klu)\begin{pmatrix}l & & m & \\ & p & & x\end{pmatrix}$.

(iii) $(ijk)\,i_x j_x\begin{pmatrix}k & & l\\ & p & \end{pmatrix}(lmu)\,m_x$. (iv) $(ijk)\,i_x j_x\begin{pmatrix}k & & l & & m & \\ & p & & q & & x\end{pmatrix}$.

The form (i) is symmetric in $k$ and $m$, by $[g_1, m_x]$, and skew in $i$ and $k$, by $[g_2, i_x]$. Hence it is reducible. The form (ii) is symmetric in $l$ and $m$, by $[g_2, m_x]$, and skew in $k$ and $l$, by $[g_1, l_p]$. Hence it is reducible. The form (iii) is reducible to a form of type (ii) by $[g_1, m_x]$ and is thus reducible. Finally, in the reduction of the form $(iku)\begin{pmatrix} k & l & m \\ p & q & x \end{pmatrix} i_x$ given in § 18, we may replace $u$ by $jj_x$, and thus reduce the form (iv). These reductions apply equally to the dual forms.

35. The other forms $C_4$ are those in which each factor $\{ab\}$ is of type $(abu)$ or $a_p b_p$. We consider these in turn, according to the number of factors $(abu)$. If there are three such factors the form is

$$(ijk)(iju)(klu)(lmu)\, m_x. \tag{35·1}$$

The form (35·1) is clearly symmetric in $i$ and $j$. By $[g_3, m_x]$ it is skew in $l, m$; by $[g_4, k \text{ of } g_1]$ it is skew in $k, l$. Hence it is skew in any pair of $k, l, m$. We may suppose, without loss of generality, that the symbols are such that $ijklm$ is a positive permutation of 01234, and denote the form by $[ij]$, so that $[ij] \equiv -[ji]$. By $[g_2, k \text{ of } g_3]$, $[ij]+[jk]+[ki] \equiv 0$. Whence all the forms are expressible in terms of the four [01], [02], [03], [04].

The dual form is reducible, since, with an analogous notation,

$$\begin{aligned} 0 &\equiv (\mathbf{ij}x)\, k_i k_j \,.\, (\mathbf{lm}x)\, k'_x k'_l u_m \equiv [(\mathbf{lj}x)\, k'_i + (\mathbf{il}x)\, k'_j]\, (\mathbf{lm}x)\, k_i k_j k'_x u_m \\ &\equiv -\tfrac{1}{2}[-(\mathbf{kij})\,(\mathbf{ki}x)\,(\mathbf{jl}x)\,(\mathbf{lm}x)\, u_m + (\mathbf{kji})\,(\mathbf{kj}x)\,(\mathbf{il}x)\,(\mathbf{lm}x)\, u_m] \\ &\equiv \tfrac{1}{2}([ki]-[jk]), \end{aligned}$$

and hence $[ki] \equiv [jk] \equiv [ij] \equiv \tfrac{1}{3}([ij]+[jk]+[ki]) \equiv 0.$

So we obtain the irreducible set

[56] $(012)(01u)(23u)(34u)\, 4_x$ Orders (1, 3). Degrees $\{1^5\}$. Four forms.

36. If two of the factors $\{ab\}$ are of type $(abu)$ the forms to consider are

$$\text{(i)} \quad (ijk)(iju)(klu)\begin{pmatrix} l & & m \\ & p & & x \end{pmatrix},$$

$$\text{(ii)} \quad (ijk)(iju)\begin{pmatrix} k & & l \\ & p & \end{pmatrix}(lmu)\, m_x,$$

$$\text{(iii)} \quad (ijk)\begin{pmatrix} i & & j \\ & p & \end{pmatrix}(klu)(lmu)\, m_x.$$

For each of these forms, $ijklm$ is a permutation of 01234, and to fix the ideas we shall suppose that $p = 0$.

The form (i) is symmetric in $i, j$ and, by $[g_3, m_x]$, in $l, m$. It may be denoted by $[ijk] \equiv [jik]$. If $l$ or $m$ is 0 the form reduces. Hence the only forms which arise are $[ab0] \equiv [ba0]$ and $[0ab] \equiv [a0b]$, where $a, b$ belong to the set 1, 2, 3, 4.

It was shown in (19·5) that

$$(klu)\begin{pmatrix} l & & m \\ & p & & x \end{pmatrix}\begin{pmatrix} k & & u \\ & q & \end{pmatrix} \equiv (klu)\begin{pmatrix} l & & m \\ & q & & x \end{pmatrix}\begin{pmatrix} k & & u \\ & p & \end{pmatrix} \tag{36·1}$$

by an argument which did not involve the separation of $p, q$ into convolutions of symbols cogredient with $k$. Hence this equivalence remains true if $q$ is replaced by $(ij)$. Taking

$p = 0$, $k = 0$, $q = (ij)$ in (36·1) the right-hand member is reducible and the left-hand member is $[ij0]$. Hence $[ij0] \equiv 0$. We are thus left with the forms $[0ab] \equiv [a0b] \equiv [ab]$, say. Taking $i = 0$ in (i) and applying respectively $[g_2, k \text{ of } g_3]$ and $[g_1, l_0]$ we obtain, since $[jk0] \equiv 0$,

$$[jk]+[kj] \equiv 0, \quad [jk]+[jl] \equiv 0$$

from which it is easily seen that all the forms (i) reduce.

Next consider the forms (ii), with $p = 0$. If $k$ or $l$ is 0 the form clearly reduces. Now, by (19·6),

$$2(lmu)\begin{pmatrix} l & & k & & u \\ & p & & q & \end{pmatrix} m_x \equiv (mlu)\begin{pmatrix} m & & k \\ & p & & x \end{pmatrix}\begin{pmatrix} l & & u \\ & q & \end{pmatrix} + (mku)\begin{pmatrix} m & & l \\ & p & & x \end{pmatrix}\begin{pmatrix} k & & u \\ & q & \end{pmatrix},$$

and in this equivalence $q$ may be replaced by $(ij)$. If $m = 0$, $p = 0$ the right-hand member is reducible, and the left-hand member is twice the form (ii) with $m = 0$. Hence the only forms (ii) which remain are those for which $i$ or $j$ is 0, and $i, j$ are clearly interchangeable. But, when $i = 0$, the form is skew in $j, k$, by $[g_2, k_0]$ and is symmetrical in $k, l$ by $[g_3, k \text{ of } g_1]$. Hence (ii) is always reducible.

The form (iii) is clearly skew in $i, j$ and is reducible if $i$ or $j$ is 0. By $[g_3, k \text{ of } g_1]$ it is skew in $k, l$; by $[g_2, m_x]$ it is skew in $l, m$. Hence it is skew in any pair of $k, l, m$. Since one of these three symbols must be 0 the form with $k = 0$ is typical. But, by $[g_2, i_0]$,

$$(ij0)\begin{pmatrix} i & & j \\ & 0 & \end{pmatrix}(0lu)\,(lmu)\,m_x \equiv (ij0)\,(0iu)\begin{pmatrix} j & & l \\ & 0 & \end{pmatrix}(lmu)\,m_x + (ij0)\begin{pmatrix} u & & j \\ & 0 & \end{pmatrix}(0li)\,(lmu)\,m_x.$$

The first term on the right is of type (ii), and so reducible. The second, by $[g_1, u_0]$, is equivalent to

$$(j0u)\begin{pmatrix} i & & j \\ & 0 & \end{pmatrix}(0li)\,(lmu)\,m_x \equiv (j0u)\begin{pmatrix} i & & l \\ & 0 & \end{pmatrix}(0ji)\,(lmu)\,m_x, \quad \text{by } [g_2, j_0],$$

and this again is of type (ii) and so reducible.

Hence the types considered give rise to no new forms. The same argument applies to the duals.

37. If there is only one bracket of type $(abu)$ the forms to consider are

$$\text{(i)} \quad (ijk)\,(iju)\begin{pmatrix} k & & l & & m & \\ & p & & q & & x \end{pmatrix},$$

$$\text{(ii)} \quad (ijk)\begin{pmatrix} i & & j \\ & p & \end{pmatrix}(klu)\begin{pmatrix} l & & m & \\ & q & & x \end{pmatrix},$$

$$\text{(iii)} \quad (ijk)\begin{pmatrix} i & & j \\ & p & \end{pmatrix}\begin{pmatrix} k & & l \\ & q & \end{pmatrix}(lmu)\,m_x.$$

If $p = q$ the form (i) is clearly reducible, the form (ii) reduces by $[g_1, m_q]$, and the form (iii) by $[g_1, l_q]$. So we may suppose $p \neq q$. The symbols $ijklm$ are 01234 in some order, and to fix the ideas we suppose $p, q$ to be 0, 1 in some order. The following forms arise for consideration:

$$\left.\begin{aligned}
[ijklm]_1 &\equiv (ijk)\,(iju)\begin{pmatrix} k & & l & & m & \\ & 0 & & 1 & & x \end{pmatrix}, & [ijklm]'_1 &\equiv (ijk)\,(iju)\begin{pmatrix} k & & l & & m & \\ & 1 & & 0 & & x \end{pmatrix},\\
[ijklm]_2 &\equiv (ijk)\begin{pmatrix} i & & j \\ & 0 & \end{pmatrix}(klu)\begin{pmatrix} l & & m & \\ & 1 & & x \end{pmatrix}, & [ijklm]'_2 &\equiv (ijk)\begin{pmatrix} i & & j \\ & 1 & \end{pmatrix}(klu)\begin{pmatrix} l & & m & \\ & 0 & & x \end{pmatrix},\\
[ijklm]_3 &\equiv (ijk)\begin{pmatrix} i & & j \\ & 0 & \end{pmatrix}\begin{pmatrix} k & & l \\ & 1 & \end{pmatrix}(lmu)\,m_x, & [ijklm]'_3 &\equiv (ijk)\begin{pmatrix} i & & j \\ & 1 & \end{pmatrix}\begin{pmatrix} k & & l \\ & 0 & \end{pmatrix}(lmu)\,m_x.
\end{aligned}\right\} \quad (37·1)$$

We refer to these sets of forms as the sets (1), (1)′, ..., (3)′ respectively.

The form $[ijklm]_1$ is symmetrical in $i, j$ and is clearly reducible if $k$ or $l$ is 0, or if $l$ or $m$ is 1. From (10·6) it follows that $\begin{matrix} & m & l & 1 & u \\ x & 1 & 0 & p & \end{matrix} \equiv 0$, and that $p$ may be replaced by $(ij)$ without affecting the validity of the reduction. Hence $[ij1lm]_1 \equiv 0$. Thus the forms (1) all reduce unless $i$ or $j = 1$; and since $i, j$ are interchangeable we need only consider the forms for which $i = 1$. Now, by $[g_2, k_0]$,

$$[1jklm]_1 + [jk1lm_1] + [k1jlm]_1 \equiv 0.$$

The second term has been seen to be reducible, and the third is $[1kjlm]_1$. Thus $[1jklm]_1$ is skew in $j$ and $k$. But it is reducible if $k = 0$ (or $l = 0$) and hence also reducible if $j = 0$. Hence $m$ must be 0, and the surviving forms of (1) are $[1jkl0]_1$, skew in $j$ and $k$. But, replacing $p$ by $(1j)$ in the relation

$$\begin{matrix} & 0 & k & l & u \\ x & 1 & 0 & p & \end{matrix} + \begin{matrix} & 0 & l & k & u \\ x & 1 & 0 & p & \end{matrix} \equiv 0$$

$[L_1 + L_2 \equiv 0$ of (10·6)] we see that $[1jkl0]_1$ is skew in $k$ and $l$. Thus there is a single surviving form (1), namely $[1jkl0]_1 \equiv [j1kl0]_1$, which is skew in any pair of $j, k, l$. Similarly, all the forms (1)′ are reducible, with the possible exception of $[0jkl1]_1' \equiv [j0kl1]_1'$, which is skew in any pair of $j, k, l$.

Consider next the forms (2). By $[g_2, m_x]$, $[ijklm]_2$ is symmetrical in $l, m$; and it is clearly skew in $i, j$. It reduces if either $i, j = 0$ or $l, m = 1$. Thus the *a priori* possible irreducible forms (2) form the sets

$$\left.\begin{aligned} &(\alpha) \quad [1jkl0]_2 \equiv [1jk0l]_2 \equiv -[j1kl0]_2 \equiv -[j1k0l]_2, \\ &(\beta) \quad [1j0lm]_2 \equiv [1j0ml]_2 \equiv -[j10lm]_2 \equiv -[j10ml]_2, \\ &(\gamma) \quad [ij1l0]_2 \equiv [ij10l]_2 \equiv -[ji1l0]_2 \equiv -[ji10l]_2. \end{aligned}\right\} \tag{37·2}$$

Now, transforming $[ijklm]_1$ by $[g_1, l_1]$,

$$\begin{aligned} [ijklm]_1 &\equiv [(ljk)\, i_1 + (lki)\, j_1 + (lij)\, k_1]\, (iju) \begin{pmatrix} k & & l \\ & 0 & \end{pmatrix} m_1 m_x \\ &\equiv -[kljim]_2 + [lkijm]_2 + [ijlkm]_1 \\ &\equiv [lkjim]_2 + [lkijm]_2 + [ijlkm]_1. \end{aligned}$$

After a permutation of symbols, this relation is seen to be equivalent to

$$[ijklm]_2 + [ijlkm]_2 \equiv [kljim]_1 - [klijm]_1. \tag{37·3}$$

In (37·3), put $i = 1$. Then the forms on the right are reducible. Hence $[1jklm]_2$ is skew in $k, l$. But it is symmetric in $l, m$. Hence it reduces. Thus the forms ($\alpha$) and ($\beta$) of (37·2) are reducible. Again, put $k = 1$, $l = 0$ in (37·3). The terms on the right are reducible and the second term on the left is reducible. Hence $[ij10m]_2 \equiv 0$; in other words the forms ($\gamma$) of (37·2) are reducible. Thus all forms of the set (2) reduce. Finally, putting $k = 1$, $m = 0$ in (37·3), we obtain

$$0 \equiv [1lji0]_1 - [1lij0]_1 \equiv 2[1lji0],$$

by the alternating property already established. Thus $[1lji0]_1 \equiv 0$. As all the other forms (1) have been proved reducible, this shows that (1) gives no irreducible forms. Considerations of symmetry show that the sets (1)′ and (2)′ are similarly reducible.

Next, by $[g_1, l_1]$

$$[ijklm]_3 \equiv (ljk) \begin{pmatrix} k & i & j \\ 1 & 0 & \end{pmatrix} (lmu)\, m_x + (ilk) \begin{pmatrix} k & j & i \\ 1 & 0 & \end{pmatrix} (lmu)\, m_x = T_1 + T_2, \text{ say.}$$

Now, by $[g_1, m_x]$,

$$T_1 \equiv (mjk)\begin{pmatrix} k & i & j \\ & 1 & 0 \end{pmatrix}(lmu)\, l_x + (lmk)\begin{pmatrix} k & i & j \\ & 1 & 0 \end{pmatrix}(lmu)\, j_x + (ljm)\begin{pmatrix} k & i & j \\ & 1 & 0 \end{pmatrix}(lmu)\, k_x$$

$$= (mjk)\begin{pmatrix} k & i & j \\ & 1 & 0 \end{pmatrix}(lmu)\, l_x + [lmkij]'_1 - [mljik]_1$$

$$\equiv (mjk)\begin{pmatrix} k & i & j \\ & 1 & 0 \end{pmatrix}(lmu)\, l_x,$$

since the forms (1) and (1′) are reducible.

Since $T_2$ is obtained from $T_1$ by interchanging $i, j$ and changing the sign,

$$T_2 \equiv -(mik)\begin{pmatrix} k & j & i \\ & 1 & 0 \end{pmatrix}(lmu)\, l_x.$$

Thus
$$[ijklm]_3 \equiv (lmu)\, l_x \begin{pmatrix} i & j \\ & 0 \end{pmatrix} k_1[(mjk)\, i_1 - (mik)\, j_1]$$

$$\equiv (lmu)\, l_x \begin{pmatrix} i & j \\ & 0 \end{pmatrix}\begin{pmatrix} k & m \\ & 1 \end{pmatrix}(ijk) = -[ijkml]_3.$$

Hence $[ijklm]_3$ is skew in $l$ and $m$. By $[g_2,\, k \text{ of } g_1]$, it is symmetric in $k$ and $l$. Hence it is reducible. Similarly the forms (3)′ are reducible. The same reasoning applies to the dual forms.

38. If no factor of type $(abu)$ occurs, the form $C_4$ is

$$(ijk)\begin{pmatrix} i & j \\ & p \end{pmatrix}\begin{pmatrix} k & l & m & \\ & q & r & x \end{pmatrix}. \tag{38·1}$$

By $[g_1, m_r]$ this is equivalent to

$$(mjk)\begin{pmatrix} k & l & i & j \\ & q & r & p \end{pmatrix} m_x + (imk)\begin{pmatrix} k & l & j & i \\ & q & r & p \end{pmatrix} m_x. \tag{38·2}$$

But, by § 17, $(abu)\begin{pmatrix} a & c & d & b \\ & q & r & p \end{pmatrix} \equiv 0$. Polarizing with respect to $m_x$,

$$(abm)\begin{pmatrix} a & c & d & b \\ & q & r & p \end{pmatrix} m_x \equiv 0.$$

Hence each term on the left of (38·2) is reducible, and so the form (38·1) is likewise reducible. The dual form reduces similarly.

*The subsystem $C_5$*

39. Consider now the forms $(ijk)\{ij\}\{kl\}\{lm\}\{mi\}\, i_x$. If any $\{ab\}$ is $a_x b_x$ the form is reducible. If $\{kl\}$ is $(klu)$ the form can be written $(ijk)(klu)\{ij\}\{lm\}\{mi\}\, i_x$, and is reduced by $[g_2, i_x]$ followed by $[g_1, m \text{ of } \{mi\}]$. If $\{lm\}$ is $(lmu)$ the form is $(ijk)(lmu)\{ij\}\{kl\}\{mi\}\, i_x$, and is reduced the $[g_2, i_x]$ followed by $[g_1, m \text{ of } \{mi\}]$. Hence it is only necessary to consider the four types

(i) $(ijk)(iju)(miu)\begin{pmatrix} k & l & m \\ & p & q \end{pmatrix} i_x$, (ii) $(ijk)(iju)\begin{pmatrix} k & l & m & i & \\ & p & q & r & x \end{pmatrix}$,

(iii) $(ijk)(miu)\begin{pmatrix} j & i \\ p & x \end{pmatrix}\begin{pmatrix} k & l & m \\ & q & r \end{pmatrix}$, (iv) $(ijk)\begin{pmatrix} k & l & m & i & j \\ & p & q & r & s \end{pmatrix} i_x$.

In each case, $ijklm$ is a permutation of 01234. As a matter of fact all these are reducible.

40. Consider first the forms (i) and (ii). Clearly $p \neq k, l$, $q \neq l, m$ or else the form reduces. By $[g_2, k_p]$ the forms are skew in $j$, $k$; hence $p \neq j$. By $[g_2, l_q]$ each form is reducible if $q = j$. By $[g_1, l_q]$ each form is reducible if $q = k$. Hence the only possible irreducible forms are those for which $q = i$, $p = m$. The form can then be written $(ijk)\,(iju)\,\{mi\}\begin{pmatrix} k & & l & & m \\ & m & & i & \end{pmatrix} i_x$, and is seen to be reducible by applying $[g_1, m_i]$, $[g_2, m$ of $\{mi\}]$, $[g_2, k_m]$ in succession.

Consider next the form (iii). By $[g_1, l_r]$ the form is symmetric in $k$ and $l$. By $[g_1, m$ of $g_2]$ the form is unaltered (to within reducible forms) by the simultaneous interchange of $k$, $m$ and $q$, $r$. Hence, as the permutations $(kl)$ and $(km)\,(qr)$ generate the direct product of the symmetric groups of permutations on the sets of symbols $(klm)$ and $(qr)$, all forms obtained from the form (iii) by making any such permutation are equivalent. Hence $q, r \neq k, l, m$, so that they must be $i, j$ in some order. Owing to symmetry in $q$, $r$ it suffices to consider the case $q = j$, $r = i$. The form then reduces by applying the successive transformations $[g_1, m_i]$, $[g_1, l_i]$, $[g_1, j_p]$, $[g_1, k_j]$.

Finally, (iv) is seen to be reducible by polarizing the reducible form $(kju)\begin{pmatrix} k & & l & & m & & i & & j \\ & p & & q & & r & & s & \end{pmatrix}$ of § 17 with respect to $i_x$.

Thus the subsystem $C_5$ produces no irreducible forms, since the arguments of the present section apply equally to the dual forms.

*The subsystem $C_6$*

41. This system comprises the forms $(ijk)\,\{il\}\,\{lj\}\,k_x$. If both factors $\{ab\}$ are of type $a_x b_x$ the form is obviously reducible. Neglecting forms which are obtainable from others by interchange of symbols, five types arise for consideration, namely:

(i) $(ijk)\,(ilu)\,(jlu)\,k_x$, (ii) $(ijk)\,(ilu)\begin{pmatrix} j & & l \\ & p & \end{pmatrix} k_x$,

(iii) $(ijk)\begin{pmatrix} i & & l & & j \\ & p & & q & \end{pmatrix} k_x$, (iv) $(ijk)\,(ilu)\,j_x k_x l_x$,

(v) $(ijk)\begin{pmatrix} i & & l \\ & p & \end{pmatrix} j_x k_x l_x$.

The first type belongs to four quadratics. By permuting $i, j, k, l$ we obtain seven linearly independent forms (Turnbull 1910). The dual forms are reducible (Todd 1948). So we have the irreducible set

[57] $(012)\,(03u)\,(13u)\,2_x$ Orders $(1, 2)$. Degrees $\{1^4\}$. Seven forms.

The remaining forms lead to complications which will now be discussed.

42. Consider next the form

$$(ijk)\,(ilu)\begin{pmatrix} j & & l \\ & p & \end{pmatrix} k_x = [ijkl], \text{ say.} \tag{42·1}$$

Then
$$[ijkl] + [ljki] \equiv (ilu)\,k_x j_p[(ijk)\,l_p - (ljk)\,i_p] \equiv (ilu)\,(ijl)\,j_p k_p k_x$$
$$\equiv (ijl)\begin{pmatrix} j & & k \\ & p & \end{pmatrix}[(klu)\,i_x + (iku)\,l_x] = [ljik] + [ijlk],$$

and hence
$$[ijkl] - [ijlk] \equiv [ljik] - [ljki] \equiv [kjli] - [kjil], \tag{42·2}$$

the last following by symmetry. Further,

$$[ijkl]+[kjil] = (ijk)\begin{pmatrix} j & & l \\ & p & \end{pmatrix}[(ilu)\,k_x-(klu)\,i_x] \equiv (ijk)\,(iku)\begin{pmatrix} j & & l & \\ & p & & x\end{pmatrix},$$

whence, permuting $ijk$ cyclically and adding,

$$\sum_{i,j,k}[ijkl] \equiv 0, \qquad (42\cdot3)$$

the summation extending to the six permutations of $i, j, k$.

If $abcd$ is a fixed order of the symbols denoted by $i, j, k, l$ we may write $[ijkl] = [ij]$ if $(ijkl)$ is an even permutation of $(abcd)$ and $[ijkl] = [ij]'$ if it is an odd permutation. Then (42·2) gives

$$[ij]-[ij]' \equiv [lj]-[lj]' \equiv [kj]-[kj]' \equiv \lambda_j, \text{ say,}$$

and (42·3) gives $$[ij]+[jk]+[ki]+[ji]'+[ik]'+[kj'] \equiv 0.$$

Hence $$\lambda_i+\lambda_j+\lambda_k \equiv [ij]+[ji]+[jk]+[kj]+[ki]+[ik].$$

The four relations of this type express $\lambda_i, \ldots, \lambda_k$ in terms of the forms $[ij]$. Hence the forms $[ij]'$ can be expressed in terms of $[ij]$. Thus the twelve forms $[ij]$ determine the rest and may be taken to constitute the irreducible set. If $p$ is one of the symbols $a, b, c, d$, say $a$, the forms with $j = a$ or $l = a$ are all reducible. The number of irreducibles is then three, which may be taken to be $[ab]$, $[ac]$, $[ad]$ (cf. Turnbull 1910, whose choice of forms is slightly different). We thus get the irreducible sets

[58] $(012)\,(13u)\begin{pmatrix}2 & & 3\\ & 0 & \end{pmatrix}0_x$ Orders (1, 1). Degrees $\{31^3\}$. Three forms.

[59] $(123)\,(14u)\begin{pmatrix}2 & & 4\\ & 0 & \end{pmatrix}3_x$ Orders (1, 1). Degrees $\{21^4\}$. Twelve forms.

43. The duals of these forms will now be shown to be reducible. There are two stages in the reduction, the second of which is laborious.

The form under consideration is

$$[ijkl] = (\mathbf{ijk})\,(\mathbf{il}x)\begin{pmatrix} & p & \\ j & & l\end{pmatrix}u_k \qquad (43\cdot1)$$

and the relations (42·2) and (42·3) hold equally for this form. Now

$$0 \equiv (ipu)\,i_jp_j\,.\,(\mathbf{kl}x)\,i'_ki'_l \equiv i_jp_ji'_ki'_l[-i_kp_xu_l+i_lp_xu_k+i_xp_ku_l-i_xp_lu_k]$$
$$= T_1+T_2+T_3+T_4 \text{ (say).}$$

And $T_1 \equiv \tfrac{1}{2}(\mathbf{ikj})\,(\mathbf{ikl})\,p_jp_xu_l \equiv \tfrac{1}{2}[(\mathbf{kj}x)\,p_i+(\mathbf{ji}x)\,p_k]\,(\mathbf{ikl})\,p_ju_l \equiv -\tfrac{1}{2}([kilj]+[iklj])$,

$T_2$ (by skew symmetry in $k, l$) $\equiv \tfrac{1}{2}([likj]+[ilkj])$,

$T_3 = i_xi_lu_l\,.\,i'_ji'_kp_jp_k+(\mathbf{ijl})\,i_xi'_ku_lp_jp_k \equiv \tfrac{1}{2}(\mathbf{ijl})\,(\mathbf{i}x\mathbf{k})\,u_lp_jp_k \equiv -\tfrac{1}{2}[ijlk]$,

$T_4$ (by skew symmetry in $k, l$) $\equiv \tfrac{1}{2}[ijkl]$.

Hence $$[ijkl]-[ijlk]+[ilkj]-[iklj]+[likj]-[kilj] \equiv 0. \qquad (43\cdot2)$$




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Permute $i$, $k$, $l$ cyclically, add, and use [42·2], and we find that $[ijkl] \equiv [ijlk] = [ij]$, say. From (43·2) it follows that

$$\begin{aligned}[il]+[li] &\equiv [ik]+[ki]\\ &\equiv [kl]+[lk] \quad \text{by symmetry}\\ &\equiv 0\end{aligned}$$

on adding and using (42·3). Hence $[ij]$ is skew in $i$ and $j$.

Next consider

$$\begin{aligned}0 &\equiv (jkl)\,(jkp)\,(plu)\,.\,(j'k'l')\,j'_i k'_x l'_i\\ &\equiv (jkl)\,(plu)\,[(l'kp)\,(j'k'j)+(jl'p)\,(j'k'k)+(jkl')\,(j'k'p)]\,j'_i k'_x l'_i\\ &= T_1+T_2+T_3, \text{ say.}\end{aligned}$$

$$\begin{aligned}T_1 &\equiv \tfrac{1}{2}k'_j(\mathbf{j}\ kl\ \mathbf{i})\,(plu)\,(l'kp)\,k'_x l'_i = \tfrac{1}{2}k'_j[k_i l_j - k_j l_i]\,(plu)\,(l'kp)\,k'_x l'_i\\ &\equiv T_{11}+T_{12}, \text{ say.}\end{aligned}$$

$$\begin{aligned}T_{11} &= \tfrac{1}{2}(plu)\,(pl'k)\,l_j l'_i[k_x k'_i k'_j + (\mathbf{ki}x)\,k'_j]\\ &= \tfrac{1}{2}[(pl'u)\,(pl'k)\,k_x\,.\,l_i l_j k'_i k'_j + (pl'k)\,k_x k'_i k'_j l_j(\mathbf{1}\ up\ \mathbf{i}) + (plu)\,(pl'k)\,l_j l'_i k'_j(\mathbf{ki}x)]\\ &\equiv \tfrac{1}{4}[(\mathbf{1}\ up\ \mathbf{i})\,(\mathbf{lj}\ kp)\,k_x k'_i k'_j + (plu)\,(\mathbf{k}\ pl'\ \mathbf{j})\,(\mathbf{ki}x)\,l_j l'_i]\\ &\equiv \tfrac{1}{4}[(u_i p_l - u_l p_i)\,(k_l p_j - k_j p_l)\,k_x k'_i k'_j - (plu)\,(\mathbf{ki}x)\,p_k l'_j l_j l'_i]\\ &\equiv \tfrac{1}{4}[\{u_i k_i k_x\,.\,p_j p_l k'_j k'_l + u_i p_j p_l k_x k'_j(\mathbf{kli})\} - \tfrac{1}{2}(\mathbf{kj}x)\,(\mathbf{kji})\,p_i p_l u_l + \tfrac{1}{2}(\mathbf{ki}x)\,(\mathbf{lji})\,(\mathbf{lj}\ up)\,p_k]\\ &\equiv \tfrac{1}{8}[-(\mathbf{kli})\,(\mathbf{kj}x)\,p_j p_l u_i - (\mathbf{kji})\,\{(\mathbf{lj}x)\,u_k + (\mathbf{kl}x)\,u_j\}\,p_i p_l + (\mathbf{ki}x)\,(\mathbf{lji})\,(u_l p_j - u_j p_l)\,p_k]\\ &\equiv \tfrac{1}{8}(-[klij]+[jikl]+[kijl]+[ijlk]+[iljk])\\ &\equiv \tfrac{1}{8}(-[kl]+[ji]+[ki]+[ij]+[il]) \equiv \tfrac{1}{8}(-[kl]+[ki]+[il]),\end{aligned}$$

since $[ij]+[ji]\equiv 0$.

$$\begin{aligned}T_{12} &\equiv \tfrac{1}{4}(\mathbf{li}\ up)\,(\mathbf{li}\ kp)\,k_j k'_j k'_x \equiv -\tfrac{1}{4}p_l p_i(u_l k_i + u_i k_l)\,k_j k'_j k'_x\\ &\equiv \tfrac{1}{8}[(\mathbf{kji})\,(\mathbf{kj}x)\,p_l p_i u_l + (\mathbf{kjl})\,(\mathbf{kj}x)\,p_l p_i u_i]\\ &\equiv \tfrac{1}{8}[(\mathbf{kji})\,\{(\mathbf{lj}x)\,u_k + (\mathbf{kl}x)\,u_j\}\,p_l p_i + (\mathbf{kjl})\,\{(\mathbf{ij}x)\,u_k + (\mathbf{ki}x)\,u_j\}\,p_l p_i]\\ &\equiv \tfrac{1}{8}(-[jikl]-[kijl]-[jlki]-[klji]) \equiv \tfrac{1}{8}(-[ji]-[ki]-[jl]-[kl]).\end{aligned}$$

Again,
$$\begin{aligned}T_2 &\equiv -\tfrac{1}{2}j'_k(\mathbf{k}\ lj\ x)\,(plu)\,(jl'p)\,j'_i l'_i \equiv -\tfrac{1}{2}(plu)\,(jl'p)\,[l_x j_k - l_k j_x]\,j'_k j'_i l'_i\\ &= T_{21}+T_{22}, \text{ say.}\end{aligned}$$

Now
$$\begin{aligned}T_{21} &\equiv \tfrac{1}{4}(\mathbf{jk}\ l'p)\,(\mathbf{jki})\,(plu)\,l'_i l_x\\ &= \tfrac{1}{4}l'_i l_x[l'_j p_k - l'_k p_j]\,[p_j l_k u_i - p_j l_i u_k + p_k l_i u_j - p_k l_j u_i + p_i l_j u_k - p_i l_k u_j].\end{aligned}$$

We denote the twelve terms in this product by $t_1 \ldots t_6$, $t'_1 \ldots t'_6$, the first six arising from the factor $l'_j p_k$ and the last six from $l'_k p_j$. We observe that the last six terms arise by interchanging $j$ and $k$ in the first six, the pairing of terms being $t_1, t'_4$; $t_2, t'_3$; $t_3, t'_2$; $t_4, t'_1$; $t_5$ $t'_6$; $t_6, t'_5$. Now

$t_1 \equiv \tfrac{1}{4}[l_i l_x u_i\,.\,p_j p_k l'_j l'_k + p_j p_k u_i l_x l'_j(\mathbf{lki})] \equiv \tfrac{1}{8}(\mathbf{lki})\,(\mathbf{l}x\mathbf{j})\,p_j p_k u_i \equiv -\tfrac{1}{8}[lk]$,

$t_2 \equiv \tfrac{1}{8}(\mathbf{lij})\,(\mathbf{li}x)\,p_k p_j u_k \equiv \tfrac{1}{8}(\mathbf{lij})\,[(\mathbf{ki}x)\,u_l + (\mathbf{lk}x)\,u_i]\,p_k p_j \equiv \tfrac{1}{8}(-[ij]-[lj])$,

$t_3 \equiv t_4 \equiv 0$,

$t_5 \equiv -\tfrac{1}{8}(\mathbf{lj}x)\,(\mathbf{lji})\,p_i p_k u_k \equiv \tfrac{1}{8}([ji]+[li])$, by interchanging $i$ and $j$ in $t_2$, and reversing the sign,

$t_6 \equiv \tfrac{1}{8}[lk]$, by interchanging $i, j$ in $t_1$ and reversing the sign.



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Hence $t_1+\ldots+t_6 \equiv \frac{1}{8}(-[ij]-[lj]+[ji]+[li])$. By interchanging $j$ and $k$,

$$t'_1+\ldots+t'_6 \equiv \frac{1}{8}(-[ik]-[lk]+[ki]+[li]).$$

Adding, and using the fact that $[ji] \equiv -[ij]$, we obtain

$$T_{21} \equiv \frac{1}{8}(2[li]-[lj]-[lk]-2[ij]-2[ik]).$$

Next,
$$\begin{aligned} T_{22} &= \tfrac{1}{2}[l'_i l'_k j'_i j'_k . (plu)(jlp)\, j_x + (plu)\, j_x j'_i j'_k l'_i (\mathbf{1k}\, pj)] \\ &\equiv \tfrac{1}{4}(\mathbf{1}\, up\, \mathbf{i})(\mathbf{1k}\, pj) j_x j'_i j'_k \equiv \tfrac{1}{4}[u_i p_l - u_l p_i][p_l j_k - p_k j_l] j_x j'_i j'_k \\ &\equiv -\tfrac{1}{4}[u_i p_k p_l j_l j_x j'_i j'_k + u_l p_i p_l j_k j_x j'_i j'_k] \\ &\equiv -\tfrac{1}{4}[u_i j_i j_x . p_k p_l j'_k j'_l + u_i p_k p_l j_x j'_k (\mathbf{jli}) - \tfrac{1}{2}(\mathbf{jki})(\mathbf{jk}x)\, p_i p_l u_l] \\ &\equiv -\tfrac{1}{8}[-(\mathbf{jli})(\mathbf{jk}x)\, p_k p_l u_i - (\mathbf{jki})\{(\mathbf{1k}x)\, u_j + (\mathbf{j1}x)\, u_k\} p_i p_l] \\ &\equiv \tfrac{1}{8}([jl]-[ki]-[ji]). \end{aligned}$$

Further,
$$\begin{aligned} T_3 &\equiv -\tfrac{1}{2}(\mathbf{1}\, jk\, up)(\mathbf{1}\, jk\, \mathbf{i})(j'k'p)\, j'_i k'_x \equiv \tfrac{1}{2} j_i k_l [(jup)\, k_i + (kup)\, j_i](j'k'p)\, j'_i k'_x \\ &= T_{31} + T_{32}, \text{ say.} \end{aligned}$$

$$\begin{aligned} T_{31} &= \tfrac{1}{2}[(jup)(jk'p)\, k'_x . j_l j'_i k_l k_i + (jup)\, k'_x j'_i k_l k_i (\mathbf{j1}\, k'p)] \\ &\equiv \tfrac{1}{4}(\mathbf{j}\, up\, \mathbf{i})(\mathbf{j1}\, k'p)\, k'_x k_l k_i \equiv \tfrac{1}{4}[u_i p_j - u_j p_i][k'_j p_l - k'_l p_j]\, k'_x k_l k_i \\ &\equiv \tfrac{1}{4}[\{u_i k_i k_x . p_j p_l k'_j k'_l + u_i p_j p_l k_i k'_j (\mathbf{kl}x)\} + u_j p_i p_j k_i k_l k'_l k'_x] \\ &\equiv \tfrac{1}{8}[(\mathbf{kij})(\mathbf{kl}x)\, p_j p_l u_i - (\mathbf{kli})(\mathbf{kl}x)\, p_i p_j u_j] \\ &\equiv \tfrac{1}{8}[(\mathbf{kij})(\mathbf{kl}x)\, p_j p_l u_i - (\mathbf{kli})\{(\mathbf{jl}x)\, u_k + (\mathbf{kj}x)\, u_l\} p_i p_j] \\ &\equiv \tfrac{1}{8}[-kj+li+ki]. \end{aligned}$$

And
$$\begin{aligned} T_{32} &\equiv -\tfrac{1}{4}(\mathbf{jil})(\mathbf{ji}\, k'p)(kup)\, k_l k'_x \\ &\equiv -\tfrac{1}{4} k_l k'_x [k'_j p_i - k'_i p_j][k_j u_i p_l - k_j u_l p_i + k_i u_l p_j - k_i u_j p_l]. \end{aligned}$$

As in the reduction of $T_{21}$ above, we write this as $(t_1+t_2+t_3+t_4)+(t'_1+t'_2+t'_3+t'_4)$, the last four terms being obtained by interchanging $i$ and $j$ in the first four. And

$$\begin{aligned} t_1 &\equiv \tfrac{1}{8}(\mathbf{kjl})(\mathbf{kj}x)\, p_i p_l u_i \equiv \tfrac{1}{8}(\mathbf{kjl})[(\mathbf{ij}x)\, u_k + (\mathbf{ki}x)\, u_j]\, p_i p_l \equiv \tfrac{1}{8}(-[jl]-[kl]), \\ t_2 &\equiv 0, \\ t_3 &\equiv -\tfrac{1}{4}[k_l k_x u_l . p_i p_j k'_i k'_j + u_l p_i p_j k_l k'_j (\mathbf{ki}x)] \equiv -\tfrac{1}{8}(\mathbf{klj})(\mathbf{ki}x)\, p_i p_j u_l \equiv \tfrac{1}{8}[kj], \\ t_4 &\equiv 0. \end{aligned}$$

Thus $(t_1+t_2+t_3+t_4) \equiv \frac{1}{8}([kj]-[jl]-[kl])$. Interchanging $i$ and $j$,

$$(t'_1+t'_2+t'_3+t'_4) \equiv \tfrac{1}{8}([ki]-[il]-[kl]),$$

and hence
$$T_{32} \equiv \tfrac{1}{8}([ki]+[kj]-[il]-[jl]-2[kl]).$$

But
$$T_{11}+T_{12}+T_{21}+T_{22}+T_{31}+T_{32} = T_1+T_2+T_3 \equiv 0.$$

Substituting and using the relations $[ab]+[ba] \equiv 0$ we obtain

$$-\tfrac{3}{8}([ik]+[il]+[kl]) \equiv 0.$$

Hence
$$[ki]+[li] \equiv [kl].$$

But the left-hand member is symmetrical in $k$ and $l$, while the right-hand member is skew. Hence $[kl] \equiv 0$ and all the dual forms are reducible.

44. Next, consider the form $(ijk)\begin{pmatrix} i & & l & & j \\ & p & & q & \end{pmatrix} k_x$. The sets $ijkl$ and $pq$ contain at least one common symbol. Let this be 0. Then the forms which arise form the sets

$$[jkl] = (0jk)\begin{pmatrix} 0 & & l & & j \\ & p & & 0 & \end{pmatrix} k_x, \quad [jkl]' = (0jk)\begin{pmatrix} j & & l & & k \\ & p & & 0 & \end{pmatrix} 0_x. \tag{44·1}$$

By $[g_1, l_0]$,
$$[jkl]' \equiv [lkj]'. \tag{44·2}$$

Again
$$[jkl]' - [jlk]' = j_p 0_x \begin{pmatrix} l & & k \\ & 0 & \end{pmatrix} [(0jk)\, l_p - (0jl)\, k_p] \equiv j_p 0_x \begin{pmatrix} l & & k \\ & 0 & \end{pmatrix} (ljk)\, 0_p$$
$$\equiv \begin{pmatrix} l & & k \\ & 0 & \end{pmatrix}\begin{pmatrix} j & & 0 \\ & p & \end{pmatrix} [(0jk)\, l_x + (0kl)\, j_x + (0lj)\, k_x].$$

Transforming the three terms on the right by $[g_1, l_0]$, $[g_1, j_p]$, $[g_1, k_0]$ respectively we obtain
$$[jkl]' - [jlk]' \equiv -[klj] - [ljk] + [kjl] + [lkj]. \tag{44·3}$$

And
$$0 \equiv (0lk)^2 . (\mathbf{p0}x)\, j_p j_0 \equiv (0lk)\, j_p j_0 [0_p l_0 k_x - 0_p l_x k_0 + 0_x l_p k_0 - 0_x l_0 k_p]$$
$$\equiv [lkj] + [klj] + [lkj]' + [klj]'.$$

Hence, by (44·2),
$$[jkl]' + [jlk]' \equiv -[lkj] - [klj]. \tag{44·4}$$

From (44·3) and (44·4), by addition
$$2[jkl]' \equiv [kjl] - [ljk] - 2[klj] \tag{44·5}$$

so that the forms $[abc]'$ are expressible in terms of $[abc]$, and from (44·2) and (44·5) the latter forms satisfy
$$[kjl] - [ljk] - 2[klj] \equiv [klj] - [jlk] - 2[kjl]$$

or
$$[ljk] - [jlk] \equiv 3([kjl] - [klj]). \tag{44·6}$$

The equivalence (44·6) gives three linearly independent relations between the six forms $[abc]$, so that if $p$ is distinct from $j$, $k$, $l$ there are three independent forms. If $p$ belongs to the set $jkl$ the form belongs to four quadratics, and it is easily seen that only one independent form survives (Turnbull 1910). So we get the irreducible sets

[60] $(012)\begin{pmatrix} 0 & & 3 & & 2 \\ & 1 & & 0 & \end{pmatrix} 1_x$ Orders $(1, 0)$. Degrees $\{3^2 1^2\}$. One form.

[61] $(023)\begin{pmatrix} 0 & & 4 & & 2 \\ & 1 & & 0 & \end{pmatrix} 3_x$ Orders $(1, 0)$. Degrees $\{321^3\}$. Three forms.

45. The dual forms reduce. For, if
$$[jkl] = (\mathbf{0jk})\begin{pmatrix} & p & & 0 & \\ 0 & & l & & j \end{pmatrix} u_k,$$
$$[jkl]' = (\mathbf{0jk})\begin{pmatrix} & p & & 0 & \\ j & & l & & k \end{pmatrix} u_0,$$

then (44·5) and (44·6) hold. And
$$0 \equiv (\mathbf{0jk})\, l_0 l_j u_k . (l'p0)^2$$
$$\equiv (l'p0)\, l_0 l_j u_k [l'_0 p_j 0_k - l'_0 p_k 0_j + p_0 0_j l'_k - p_0 0_k l'_j]$$
$$= T_1 + T_2 + T_3 + T_4, \text{ say.}$$

Now

$$T_1 \equiv -\tfrac{1}{2}(\mathbf{10j})\,(\mathbf{10}\,p0)\,u_k p_j 0_k \equiv \tfrac{1}{2}(\mathbf{10j})\,p_0 0_l u_k p_j 0_k \equiv \tfrac{1}{2}(\mathbf{10k})\,p_0 0_l u_k p_j 0_j, \text{ by } [g_1, 0_k]$$
$$\equiv -\tfrac{1}{2}[lkj],$$

$$T_2 \equiv \tfrac{1}{2}(\mathbf{10j})\,(\mathbf{10}\,p0)\,u_k p_k 0_j \equiv -\tfrac{1}{2}(\mathbf{10j})\,p_0 0_l u_k p_k 0_j \equiv -\tfrac{1}{2}[(\mathbf{k0j})\,p_l + (\mathbf{10k})\,p_j]\,p_0 0_l u_k 0_j$$
$$\equiv \tfrac{1}{2}(-[jkl]+[lkj]),$$

$$T_3 \equiv [(l'u0)\,p_k + (l'pu)\,0_k]\,l_0 l_j p_0 0_j l'_k = T_{31} + T_{32}, \text{ say.}$$

$$T_{31} \equiv (l'u0)\,0_j l'_j\,.\,p_0 p_k l_0 l_k + (l'u0)\,0_j p_0 p_k l_0 (\mathbf{ljk}) \equiv \tfrac{1}{2}(\mathbf{ljk})\,(\mathbf{10}\,u0)\,0_j p_0 p_k$$
$$\equiv -\tfrac{1}{2}(\mathbf{ljk})\,u_0 0_l 0_j p_0 p_k \equiv -\tfrac{1}{2}[(\mathbf{0jk})\,p_l + (\mathbf{10k})\,p_j]\,u_0 0_l 0_j p_k, \text{ by } [g_1, p_0]$$
$$\equiv \tfrac{1}{2}([kjl]' - [klj]') \equiv \tfrac{1}{2}([jkl] - [lkj] - [jlk] + [ljk]), \quad \text{by } (44\cdot3),$$

$$T_{32} \equiv (l'pu)\,l'_0 p_0\,.\,0_j 0_k l_j l_k + (l'pu)\,p_0 0_j 0_k l_j (\mathbf{10k}) \equiv \tfrac{1}{2}(\mathbf{10k})\,(\mathbf{lj}\,pu)\,p_0 0_j 0_k$$
$$\equiv \tfrac{1}{2}(\mathbf{10k})\,[p_l u_j - p_j u_l]\,p_0 0_j 0_k \equiv \tfrac{1}{2}[(\mathbf{j0k})\,p_l u_j p_0 0_l 0_k - (\mathbf{10k})\,p_j u_l p_0 0_j 0_k]$$
$$\equiv \tfrac{1}{2}([kjl] - [klj]),$$

$$T_4 \equiv \tfrac{1}{2}(\mathbf{lj0})\,(\mathbf{lj}\,p0)\,u_k p_0 0_k \equiv \tfrac{1}{2}(\mathbf{lj0})\,[p_l 0_j - p_j 0_l]\,u_k p_0 0_k$$
$$\equiv \tfrac{1}{2}[(\mathbf{kj0})\,p_l 0_j u_k p_0 0_l - (\mathbf{lk0})\,p_j 0_l u_k p_0 0_j], \text{ by } [g_1, 0_k]$$
$$\equiv \tfrac{1}{2}(-[jkl] - [lkj]).$$

Hence, adding and doubling,

$$-[jkl] - [jlk] + [kjl] - [klj] - 2[lkj] + [ljk] \equiv 0. \tag{45·1}$$

Interchanging $j$, $l$ and adding, $\quad [jkl] + [lkj] \equiv 0.$

Whence, from (45·1), $[lkj] \equiv 0$. Hence the form is reducible.

46. The next type, $(ijk)\,(ilu)\,j_x k_x l_x$ is reducible, and belongs to four quadratics (Turnbull 1910); the reduction applying to the dual form as well. So there remains for consideration the form $(ijk)\begin{pmatrix} i & & l \\ & p & \end{pmatrix} j_x k_x l_x$. This is skew in $j$, $k$. By $[g_1, l_x]$ it is symmetric in $i$, $l$. By $[g_1, l_p]$ the sum of the three forms obtained by cyclic permutation of $j$, $k$, $l$ is reducible. The form may be denoted by $[jk] \equiv -[kj]$, and the relations $[jk] + [kl] + [lj] \equiv 0$ show that all the forms can be expressed in terms of three, e.g. $[ji]$, $[jk]$, $[jl]$. If $p$ belongs to the set $ijkl$ the form belongs to four quadratics, and there is one irreducible form (Turnbull 1910). The irreducible sets are

[62] $\quad (012)\begin{pmatrix} 1 & & 3 \\ & 0 & \end{pmatrix} 0_x 2_x 3_x \quad$ Orders $(3, 0)$. $\quad$ Degrees $\{31^3\}$. $\quad$ One form.

[63] $\quad (123)\begin{pmatrix} 1 & & 4 \\ & 0 & \end{pmatrix} 2_x 3_x 4_x \quad$ Orders $(3, 0)$. $\quad$ Degrees $\{21^4\}$. $\quad$ Three forms.

The dual forms reduce. For, using the corresponding notation to denote these forms,

$$0 \equiv (piu)^2\,.\,(\mathbf{jkl})\,i'_j i'_k u_l$$
$$\equiv (piu)\,[u_j p_k i_l - u_j p_l i_k + u_k p_l i_j - u_k p_j i_l]\,i'_j i'_k u_l = T_1 + T_2 + T_3 + T_4, \text{ say.}$$

$$T_1 = (piu)\,p_k i_k\,.\,u_j u_l i'_j i'_l + (piu)\,p_k u_j u_l i'_j (\mathbf{ilk}) \equiv \tfrac{1}{2}(\mathbf{ilk})\,(\mathbf{i}\,up\,\mathbf{j})\,p_k u_j u_l$$
$$\equiv -\tfrac{1}{2}(\mathbf{ilk})\,p_j p_k u_i u_j u_l \equiv -\tfrac{1}{2}[il],$$

$$T_2 \equiv \tfrac{1}{2}(\mathbf{ikj})\,(\mathbf{ik}\,up)\,u_j u_l p_l \equiv \tfrac{1}{2}(\mathbf{ikj})\,[u_i p_k - u_k p_i]\,u_j u_l p_l \equiv \tfrac{1}{2}([ji] - [kj])$$

and, by skew symmetry in $j$ and $k$,

$$T_3 \equiv \tfrac{1}{2}(-[ki]+[jk]), \quad T_4 \equiv \tfrac{1}{2}[il].$$

Hence $\qquad 0 \equiv [ji]-[kj]-[ki]+[jk] \equiv [ji]+[jk]+[ik]+[jk] \equiv 3[jk],$

since $[ji]+[ik] \equiv -[kj] \equiv [jk]$, and the form reduces.

*The subsystem $C_7$*

47. This system comprises the forms $(ijk)\{il\}\{lj\}\{km\}m_x$. We shall prove that all these forms can be expressed in terms of forms $C_4$, and all equivalence relations in the present section are to be interpreted in this sense; i.e. $f \equiv g$ will mean $f-g$ is expressible in terms of forms $C_4$ and reducible forms. All reductions apply equally to the dual forms.

The form is clearly reducible if $\{km\}$ is $k_x m_x$, or if $\{il\}$, $\{lj\}$ are both of type $a_x b_x$. Consider first the forms in which $\{km\}$ is $(kmu)$. Since $i$ and $j$ appear in a symmetrical manner there are four types to consider, namely

$$\text{(i)}\quad (ijk)\, i_x l_x \{lj\}\,(kmu)\, m_x, \qquad \text{(ii)}\quad (ijk)\begin{pmatrix} i & l & j \\ & p & q & \end{pmatrix}(kmu)\, m_x,$$

$$\text{(iii)}\quad (ijk)\,(ilu)\begin{pmatrix} j & & l \\ & p & \end{pmatrix}(kmu)\, m_x, \qquad \text{(iv)}\quad (ijk)\,(ilu)\,(jlu)\,(kmu)\, m_x.$$

The form (i) reduces to type $C_4$ by $[g_2, l_x]$.

Consider the form (ii). By $[g_1, l_p]$, $[g_1, l_q]$ it is symmetrical, to within forms $C_4$, in $jl$ and in $il$. Hence $p, q \neq i, j, l$ so that they are $k, m$ in some order. The case $p=k$, $q=m$ is typical. The form (ii) is then

$$(ijk)\begin{pmatrix} i & l & j \\ & k & m & \end{pmatrix}(kmu)\, m_x \equiv (mjk)\begin{pmatrix} i & l & j \\ & k & m & \end{pmatrix}(kiu)\, m_x + (imk)\begin{pmatrix} i & l & j \\ & k & m & \end{pmatrix}(kju)\, m_x.$$

The first term on the right reduces to a form $C_4$ by $[g_1, i_k]$, and the second to forms $C_4$ by $[g_2, m_x]$ followed by $[g_1, j_m]$. Hence (ii) gives no new forms.

The form (iii), to within forms $C_4$, is skew in $i$, $k$ by $[g_2, k \text{ of } g_3]$, and skew in $i$, $l$ by $[g_1, l_p]$. Hence it is skew in $k$, $l$. By $[g_1, l \text{ of } g_2]$ it is symmetric in $j$, $l$. Hence it is reducible.

The form (iv) is skew in $i, j$. To within forms $C_4$, it is skew in $j$, $l$ by $[g_1, l \text{ of } g_2]$, skew in $i$, $k$ by $[g_3, k \text{ of } g_4]$, and skew in $k$, $m$ by $[g_1, m_x]$. Hence, if the form be denoted by $[ijklm]$, $[ijklm] \equiv \pm[01234]$ according as $ijklm$ is an even or odd permutation of 01234. Hence

$$\begin{aligned} 2[ijklm] &\equiv [ijklm]-[kijml] \equiv (ijk)\,(kmu)\,(jlu)\,[(ilu)\, m_x - (imu)\, l_x] \\ &\equiv (ijk)\,(kmu)\,(jlu)\,(mlu)\, i_x \qquad\qquad (47\cdot 1) \\ &\equiv [(mjk)\,(kiu)+(imk)\,(kju)]\,(jlu)\,(mlu)\, i_x \\ &\equiv [mjkli]+2[imklj], \text{ by } (47\cdot 1), \\ &\equiv -3[ijklm] \end{aligned}$$

and hence $[ijklm] \equiv 0$.

Next suppose that $\{km\}$ is $\begin{pmatrix} k & & m \\ & p & \end{pmatrix}$. Then, by $[g_1, m_x]$, the form is symmetric in $k$, $m$. If $\{il\}$ is $(ilu)$ the form is skew in $i$, $k$ by $[g_2, k_p]$. Hence it reduces. Similarly it is reducible if $\{jl\}$ is $(jlu)$. So the only possible types are

$$\text{(v)}\quad (ijk)\begin{pmatrix} i & l & j \\ & q & r & \end{pmatrix}\begin{pmatrix} k & & m \\ & p & & x \end{pmatrix}, \qquad \text{(vi)}\quad (ijk)\begin{pmatrix} i & & l \\ & q & \end{pmatrix}\begin{pmatrix} k & & m \\ & p & \end{pmatrix} j_x l_x m_x.$$

The form (v), by $[g_1, m_p]$ is

$$(mjk)\begin{pmatrix} j & & l & & i & & k \\ & r & & q & & p & \end{pmatrix} m_x + (mki)\begin{pmatrix} k & & j & & l & & i \\ & p & & r & & q & \end{pmatrix} m_x,$$

each term of which reduces on polarizing a reducible form of type $(abu)\begin{pmatrix} a & & c & & d & & b \\ & e & & f & & g & \end{pmatrix}$. The form (vi) reduces by writing $jj_x$ for $u$ in the reducible form

$$(kiu)\begin{pmatrix} k & & m & \\ & p & & x \end{pmatrix}\begin{pmatrix} i & & l & \\ & q & & x \end{pmatrix}$$

of § 20. Hence the subsystem $C_7$ yields no new forms.

*The subsystem* $C_8$

48. This comprises the forms $(ijk)\{il\}\{lm\}\{mj\}k_x$. We shall prove that these forms are expressible in terms of forms $C_4$ and $C_7$, and equivalences in this section will be interpreted accordingly. All the reductions apply equally to the dual forms.

If $\{il\}$ is $i_x l_x$ the form can be rewritten as $(ijk)\{ik\}\{jm\}\{ml\}l_x$, of type $C_4$. Similarly, it reduces to type $C_4$ if $\{mj\}$ is $m_x j_x$. If $\{lm\}$ is $l_x m_x$ the form can be rewritten as $(ijk)\{il\}\{lk\}\{jm\}m_x$, of type $C_7$. If $\{lm\}$ is $(lmu)$ the form is $(ijk)(lmu)\{il\}\{mj\}k_x$ and reduces to type $C_7$ by $[g_2, k_x]$. Hence we need only consider the three types

$$\text{(i)}\quad (ijk)(ilu)(jmu)\begin{pmatrix} l & & m \\ & p & \end{pmatrix} k_x, \qquad \text{(ii)}\quad (ijk)(ilu)\begin{pmatrix} j & & m & & l \\ & q & & p & \end{pmatrix} k_x, \qquad \text{(iii)}\quad (ijk)\begin{pmatrix} i & & l & & m & & j \\ & r & & p & & q & \end{pmatrix} k_x.$$

The form (i), by $[g_1, l \text{ of } g_2]$, is equivalent to

$$(ilk)(iju)(jmu)\begin{pmatrix} l & & m \\ & p & \end{pmatrix} k_x + (ijl)(iku)(jmu)\begin{pmatrix} l & & m \\ & p & \end{pmatrix} k_x;$$

and the second term is of type $C_7$ while the first reduces to type $C_7$ by $[g_3, k_x]$. Hence this form can be ignored.

To within forms $C_4$, the form (ii) is skew in $i$, $l$ by $[g_1, l_p]$, symmetric in $j$, $m$ by $[g_1, m_p]$, and skew in $i$, $k$ by $[g_2, k_x]$. Hence it reduces if $p$ is $i, j, k, l, m$. Thus, for five quadratics, the form must be reducible to previous types.

Finally, (iii) is seen to be reducible by polarizing the reducible form $(iju)\begin{pmatrix} i & & l & & m & & j \\ & r & & p & & q & \end{pmatrix}$ with respect to $k_x$.

## *Subdivision of the system* D

49. These are forms $PQ$, where $P$ is a product of two factors of type $(abc)$ while $Q$ contains no such factors, and the duals of these forms. Taking the forms with brackets $(abc)$ as the typical ones, the possible forms of $P$ are $(ijk)^2$, $(ijk)(ijl)$, $(ijk)(ilm)$. In the first case $P$ is the whole form. Since if $Q$ contains a factor $u_p$ a reduction to forms C is possible, $Q$ is composed of factors of type $\{ab\}$, and it is easily seen that if $P$ is $(ijk)(ijl)$ these factors must form a generalized chain $\{k, l\}$ while if $P$ is $(ijk)(ilm)$ they form two generalized chains which are of the form $\{j, k\}\{l, m\}$ or $\{j, l\}\{k, m\}$. Neglecting obviously reducible forms and those which can be derived from given forms by permutation of symbols, the *a priori* possibilities are given in the following list; the subsystems $D_1, \ldots, D_6$ being reserved for detailed discussion





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and the others being reducible for reasons stated (Turnbull's arrangement is slightly different):

| | | |
|---|---|---|
| $D_1$ | $(ijk)^2$. | |
| $D_2$ | $(ijk)(ijl)\{kl\}$. | |
| $D_3$ | $(ijk)(ijl)\{ki\}\{il\}$. | |
| | $(ijk)(ijl)\{ki\}\{ij\}\{jl\}$ | Reducible by $[g_1, l \text{ of } \{jl\}]$. |
| | $(ijk)(ijl)\{ki\}\{im\}\{mj\}\{jl\}$ | Reducible by $[g_1, l \text{ of } \{jl\}]$. |
| $D_4$ | $(ijk)(ijl)\{km\}\{ml\}$. | |
| $D_5$ | $(ijk)(ijl)\{km\}\{mi\}\{il\}$. | |
| | $(ijk)(ijl)\{km\}\{mi\}\{ij\}\{jl\}$ | Reducible by $[g_1, l \text{ of } \{jl\}]$. |
| | $(ijk)(ilm)\{jk\}\{lm\}$ | Reducible to $D_6$ by $[g_1, l \text{ of } g_2]$. |
| | $(ijk)(ilm)\{jk\}\{li\}\{im\}$ | Reducible to $D_5$ by $[g_1, m \text{ of } \{im\}]$. |
| | $(ijk)(ilm)\{jk\}\{li\}\{ij\}\{jm\}$ | Reducible by $[g_1, m \text{ of } \{jm\}]$. |
| | $(ijk)(ilm)\{jk\}\{lj\}\{jm\}$ | Reducible to $D_5$ by $[g_2, j \text{ of } \{jm\}]$ followed by $[g_1, l \text{ of } \{lj\}]$. |
| | $(ijk)(ilm)\{ji\}\{ik\}\{li\}\{im\}$ | Reducible by $[g_1, m \text{ of } \{im\}]$. |
| | $(ijk)(ilm)\{ji\}\{ik\}\{lj\}\{jm\}$ | Reducible by $[g_1, m \text{ of } \{jm\}]$. |
| $D_6$ | $(ijk)(ilm)\{jl\}\{km\}$. | |
| | $(ijk)(ilm)\{jl\}\{ki\}\{im\}$ | Reducible to $D_5$ by $[g_1, m \text{ of } \{im\}]$. |

The subsystems $D_1, \ldots, D_6$ will now be considered in order.

*The subsystems $D_1$ and $D_2$*

50. These forms belong to four or fewer quadratics, with the exception of the form $(ijk)(ijl)\begin{pmatrix} k & & l \\ & p & \end{pmatrix}$ of $D_2$. This form is symmetric in $i$, $j$ and also in $k$, $l$, so that it may be denoted by $[ij] = [ji]$. By $[g_2, k_p]$, $[ij]+[jk]+[ki] \equiv 0$. These relations imply

$$[ij]+[ik]+[il] \equiv 0,$$

as is easily seen, so that the forms can be expressed in terms of two of them, i.e. $[ij]$, $[ik]$. The same remark applies to the dual forms. Using the known results for three and four quadratics, the subsystems $D_1$ and $D_2$ furnish the irreducible sets

| | | | | |
|---|---|---|---|---|
| [64] | $(012)^2$ | Orders $(0, 0)$. | Degrees $\{1^3\}$. | One form. |
| [65] | $(\mathbf{012})^2$ | Orders $(0, 0)$. | Degrees $\{2^3\}$. | One form. |
| [66] | $(012)(013)(23u)$ | Orders $(0, 1)$. | Degrees $\{1^4\}$. | Three forms. |
| [67] | $(\mathbf{012})(\mathbf{013})(\mathbf{23}x)$ | Orders $(1, 0)$. | Degrees $\{2^4\}$. | Three forms. |
| [68] | $(123)(124)\begin{pmatrix} 3 & & 4 \\ & 0 & \end{pmatrix}$ | Orders $(0, 0)$. | Degrees $\{21^4\}$. | Two forms. |
| [69] | $(\mathbf{012})(\mathbf{013})\begin{pmatrix} & 4 & \\ 2 & & 3 \end{pmatrix}$ | Orders $(0, 0)$. | Degrees $\{2^4 1\}$. | Two forms. |
| [70] | $(012)(013)\,2_x 3_x$ | Orders $(2, 0)$. | Degrees $\{1^4\}$. | Two forms. |

The duals of the forms [70] are reducible (Todd 1948).

*The subsystem $D_3$*

51. These are the forms $(ijk)(ijl)\{ki\}\{il\}$. Clearly $\{ki\}$ and $\{il\}$ cannot both be of type $a_x b_x$. So the possible types are

$$\text{(i)}\quad (ijk)(ijl)(kiu)(ilu), \qquad \text{(ii)}\quad (ijk)(ijl)(kiu)\begin{pmatrix} i & & l \\ & p & \end{pmatrix},$$

$$\text{(iii)}\quad (ijk)(ijl)(kiu)\,i_x l_x, \qquad \text{(iv)}\quad (ijk)(ijl)\begin{pmatrix} k & & i & & l \\ & p & & q & \end{pmatrix},$$

$$\text{(v)}\quad (ijk)(ijl)\begin{pmatrix} k & & i \\ & p & \end{pmatrix} i_x l_x.$$

The first form is symmetric in $k$, $l$ and, by $[g_2, k \text{ of } g_3]$, skew in $j$, $k$. Hence it is reducible.

52. Consider now the form (ii). By $[g_1, l_p]$ it is skew in $j$, $l$; by $[g_2, k \text{ of } g_3]$ it is skew in $j$, $k$. Hence it is skew in any pair of $j$, $k$, $l$. Thus it is reducible if $p$ is either $i$, $j$, $k$ or $l$; and if $p$ is distinct from these there is just one form $[jkl, i]$ which is skew symmetric in $j$, $k$, $l$.

Since $i$ occurs four times in the form (ii), this symbol represents a class of forms the difference between any two of which is expressible in terms of forms in which two equivalent symbols $i, i'$ have been convolved into a symbol $\mathbf{i}$. These forms are easily seen to be of tpye [24] (§ 16) and have the form $(jku)\begin{pmatrix} j & & l & & k \\ & i & & p & \end{pmatrix}$.

Now the form $[jkl, p]$ obtained by interchanging $i$ and $p$ in (ii) has the same partial degrees as $[jkl, i]$. We shall show that these forms are equivalent *modulo* forms of type [24]. In fact, all equivalences being to this modulus,

$$\begin{aligned} 2[jkl, i] &\equiv [jkl, i] - [jlk, i] \equiv (ijk)(ijl)[(ki'u)i'_p l_p - (li'u)\,i'_p k_p] \\ &\equiv (ijk)(ijl)(i'lk)(i'pp')(upp') \equiv 2(ijk)(ijl)(i'pk)(i'lp')(upp') \\ &\equiv 2(ijl)(i'pk)(i'lp')[(pjk)(uip') + (ipk)(ujp') + (ijp)(ukp')] = T_1 + T_2 + T_3, \text{ say.} \end{aligned}$$

Now $\quad T_1 \equiv 2(ijl)(i'pk)(uip')([(i'lp)(p'jk) + (\mathbf{p}\,jk\,i'l)] = T_{11} + T_{12}$, say,

where $\quad T_{11} \equiv -2(i'pk)(i'pl)(ijl)(p'jk)(ip'u) \qquad (52{\cdot}1)$

and $\quad T_{12} \equiv (\mathbf{p}\,jk\,i'l)(\mathbf{p}\,ki'\,ui)(ijl) \equiv -(i'jk)(kui)(ijl)\,i'_p l_p$

$\qquad \equiv [jkl, i], \qquad (52{\cdot}2)$

since any form in which $i, i'$ and $p, p'$ are both convolved is either reducible or expressible in terms of forms of the set [24]. Further,

$$\begin{aligned} T_2 &\equiv -(\mathbf{i}\,pk\,jl)(\mathbf{i}\,pk\,lp')(ujp') \equiv p_i k_i (pjl)(klp')(ujp') \equiv [ljk, p] \\ &\equiv [jkl, p], \end{aligned} \qquad (52{\cdot}3)$$

and

$$\begin{aligned} T_3 &\equiv 2(i'\,lp')(ijp)(ukp')[(iji')(lpk) + (ijk)(i'pl)], \text{ by } [g_2, l \text{ of } g_1], \\ &\equiv -j_i(\mathbf{i}\,jp\,lp')(ukp')(lpk) + (\mathbf{p}\,li'\,ij)(\mathbf{p}\,li'\,uk)(ijk) \\ &\equiv -j_i p_i(jlp')(ukp')(lpk) - l_p i'_p(lij)(i'uk)(ijk) \equiv -[lkj, p] - [jkl, i] \\ &\equiv [jkl, p] - [jkl, i]. \end{aligned} \qquad (52{\cdot}4)$$

From the relations (52·1) to (52·4) it follows that

$$[jkl, i]-[jkl, p] \equiv -(i'pk)(i'pl)(ijl)(p'jk)(ip'u).$$

But the left-hand member of this expression is unaltered if $k$ is interchanged with $j$ and $i$ with $p$, while the right-hand member is changed in sign if $k$ is interchanged with $l$, $i$ with $p'$ and $i'$ with $p$. Hence $[jkl, i] \equiv [jkl, p]$ *modulo* forms of the set [24]. Thus the corresponding irreducible set is

[71] $(123)(124)(31u)\begin{pmatrix}1 & & 4\\ & 0 & \end{pmatrix}$ Orders (0, 1). Degrees $\{2^2 1^3\}$. One form.

The dual form is reducible, for it is

$$(\mathbf{123})(\mathbf{124})(\mathbf{31}x)\begin{pmatrix} & 0 & \\ 1 & & 4\end{pmatrix} \equiv -2[2_1 2'_1 2_3 2'_4(\mathbf{31}x)\, 0_1 0_4] \equiv 0.$$

53. The form (iii) is skew in $j$, $k$ by $[g_2, k \text{ of } g_3]$, and is symmetric in $k$, $l$ by $[g_3, l_x]$. Hence it is reducible, and the dual form reduces in the same way.

The form (iv) is skew in $j$, $k$ by $[g_2, k_p]$, and skew in $j$, $l$ by $[g_1, l_q]$. Hence it is skew in any pair of $j$, $k$, $l$. Hence $p, q \neq i, j, k, l$, and so $p = q$ and the form is reducible. The dual form reduces similarly.

The form (v) is skew in $j$, $k$ by $[g_2, k_p]$ and skew in $j$, $l$ by $[g_1, l_x]$. Hence $p \neq i, j, k, l$; and there is one form arising for given $i$, $p$ which is skew in any pair of $j$, $k$, $l$. Another form of the same partial degrees arises by interchange of $i$ and $p$, and gives rise to the irreducible set

[72] $(123)(124)\begin{pmatrix}3 & & 1\\ & 0 & \end{pmatrix}1_x 4_x$ Orders (2, 0). Degrees $\{2^2 1^3\}$. Two forms.

The dual form is reducible since

$$(\mathbf{123})(\mathbf{124})\begin{pmatrix} & 0 & \\ 3 & & 1\end{pmatrix}u_1 u_4 \equiv -2[2_1 2'_1 2_3 2'_4\, 0_3 0_1 u_1 u_4] \equiv 0.$$

*The subsystem* $D_4$

54. These are the forms $(ijk)(ijl)\{km\}\{ml\}$. If $\{km\}$ is $(kmu)$ the form reduces to type $D_6$ by $[g_2, m \text{ of } \{ml\}]$. If $\{ml\}$ is $(mlu)$ it reduces to $D_6$ by $[g_1, m \text{ of } \{km\}]$. We shall consider here the forms where neither bracket is of type $(abu)$. Since the form is obviously reducible if $\{km\}$ and $\{ml\}$ are both of type $a_x b_x$ there are two cases to consider, namely

$$\text{(i)}\quad (ijk)(ijl)\begin{pmatrix}k & & m\\ & p & \end{pmatrix}m_x l_x, \qquad \text{(ii)}\quad (ijk)(ijl)\begin{pmatrix}k & & m & & l\\ & p & & q & \end{pmatrix}.$$

Consider then the form (i). Denote this provisionally by $[ijklm]$, and suppose that $p$ is 0. Then the form is reducible if $k, m = 0$; and $[ijklm] = [jiklm]$. By $[g_1, l_x]$

$$[ijklm]+[jlkim]+[likjm] \equiv 0. \tag{54·1}$$

Hence the forms in which $l = 0$ can be expressed in terms of those in which $i$ or $j = 0$. Since the forms are symmetric with respect to $i$ and $j$ we may confine our attention to the forms with $i = 0$, namely,

$$[jklm] = (0jk)(0jl)\begin{pmatrix}k & & m\\ & 0 & \end{pmatrix}m_x l_x. \tag{54·2}$$

By $[g_2, k_0]$ this form is skew in $j$ and $k$. By $[g_1, m_0]$ followed by $[g_2, k_0]$ it is symmetric for the simultaneous interchange of $j$, $k$ and $l$, $m$, so that it is also skew in $l$, $m$. From (54·1),

$$[jklm]+[lkjm] \equiv -(jlk)\,(jl0)\begin{pmatrix}k & & m\\ & 0 & \end{pmatrix} 0_x m_x \equiv (ljk)\,[(0ml)\,j_x+(0jm)\,l_x]\begin{pmatrix}k & & m\\ & 0 & \end{pmatrix} 0_x$$

$$\equiv (ljk)\left[(0mk)\begin{pmatrix}l & & m\\ & 0 & \end{pmatrix} j_x+(0km)\begin{pmatrix}j & & m\\ & 0 & \end{pmatrix} l_x\right]0_x, \quad \text{by } [g_2, k_0]$$

$$\equiv [(0jk)\,l_x+(0lj)\,k_x]\,(0mk)\begin{pmatrix}l & & m\\ & 0 & \end{pmatrix} j_x+[(0kl)\,j_x+(0lj)\,k_x]\,(0km)\begin{pmatrix}j & & m\\ & 0 & \end{pmatrix} l_x$$

$$\equiv [kmjl]+(0lj)\,(0mk)\begin{pmatrix}l & & m\\ & 0 & \end{pmatrix} j_x k_x+[kmlj]+(0lj)\,(0km)\begin{pmatrix}j & & m\\ & 0 & \end{pmatrix} k_x l_x$$

$$\equiv [kmjl]+(0lj)\,(0ml)\begin{pmatrix}k & & m\\ & 0 & \end{pmatrix} j_x k_x+[kmlj]+(0lj)\,(0jm)\begin{pmatrix}k & & m\\ & 0 & \end{pmatrix} k_x l_x$$

$$\equiv [kmjl]-[lmjk]+[kmlj]-[jmlk] \equiv -[lmjk]-[jmlk],$$

using the skew-symmetry property. Hence, again using this property,

$$[jklm]+[lmjk] \equiv [jmkl]+[kljm]$$
$$\equiv [jlmk]+[mkjl] \quad \text{by symmetry.}$$

In view of this relation the number of linearly independent forms is reduced from six to four, and we have the irreducible set

[73] $(012)\,(013)\begin{pmatrix}2 & & 4\\ & 0 & \end{pmatrix} 3_x 4_x$ Orders (2, 0). Degrees $\{31^4\}$. Four forms.

The dual forms are reducible. For, using the corresponding notation,

$$0 \equiv (0ju)\,0_k j_k \,.\, (\mathbf{0lm})\,j'_0 j'_l u_m \equiv [j_0 u_l 0_m+u_0 0_l j_m-u_0 0_m j_l]\,0_k j_k j'_0 j'_l u_m = T_1+T_2+T_3, \text{ say.}$$

$$T_1 \equiv -\tfrac{1}{2}(\mathbf{j0k})\,(\mathbf{j0l})\,0_k 0_m u_m u_l \equiv -\tfrac{1}{2}[jklm],$$

$$T_2 \equiv 0_k 0_l j_k j_l \,.\, u_0 u_m j'_0 j'_m+0_k 0_l j_k u_0 u_m j'_0 (\mathbf{jml}) \equiv \tfrac{1}{2}(\mathbf{jml})\,(\mathbf{jk0})\,0_k 0_l u_0 u_m$$
$$\equiv \tfrac{1}{2}[(\mathbf{0ml})\,u_j+(\mathbf{0jm})\,u_l]\,(\mathbf{0jk})\,0_k 0_l u_m \equiv \tfrac{1}{2}[(\mathbf{0ml})\,(\mathbf{0lk})\,0_k 0_j u_j u_m+(\mathbf{0jm})\,(\mathbf{0jk})\,0_k 0_l u_l u_m]$$
$$\equiv \tfrac{1}{2}(-[lkmj]+[jkml]),$$

$$T_3 \equiv \tfrac{1}{2}(\mathbf{jl0})\,(\mathbf{jlk})\,0_k 0_m u_0 u_m \equiv \tfrac{1}{2}(\mathbf{jl0})\,[(\mathbf{0lk})\,u_j+(\mathbf{0kj})\,u_l]\,0_k 0_m u_m$$
$$\equiv \tfrac{1}{2}(-[lkjm]-[jklm]).$$

Hence $$0 \equiv \tfrac{1}{2}(-[jklm]-[lkmj]+[jkml]-[lkjm]-[jklm]) \equiv -\tfrac{3}{2}[jklm],$$

by the skew-symmetry property, and so the form is reducible.

55. Consider next the form (ii), $(ijk)\,(ijl)\begin{pmatrix}k & & m & & l\\ & p & & q & \end{pmatrix}$. Here $ijklm$ is a permutation of 01234. To fix the ideas we suppose that $p = 0$, $q = 1$. The possible irreducible types are

(a) $(ij0)\,(ij1)\begin{pmatrix}1 & & m & & 0\\ & 0 & & 1 & \end{pmatrix}$,

(b) $(0ij)\,(0i1)\begin{pmatrix}1 & & m & & j\\ & 0 & & 1 & \end{pmatrix}$, and a similar type with 0, 1 interchanged,

(c) $(01k)\,(01l)\begin{pmatrix}k & & m & & l\\ & 0 & & 1 & \end{pmatrix}$.

Of these, (*a*) is expressible in terms of (*b*) by $[g_2, 0_1]$ and (*b*) in terms of (*c*) by $[g_1, 1_0]$. So it suffices to consider the form (*c*). Applying in succession $[g_1, m_0]$, $[g_2, k_0]$, $[g_1, 1_0]$ we see that the form is skew in $l$, $m$. Similarly, by $[g_2, m_1]$, $[g_1, l_1]$, $[g_2, 0_1]$ we see that it is skew in $k$, $m$. Hence the six forms (*c*) obtained by permuting $k, l, m$ are expressible in terms of one of them, and we obtain the irreducible set

$$[74] \quad (012)\,(013)\begin{pmatrix}2 & & 4 & & 3\\ & 0 & & 1 & \end{pmatrix} \quad \text{Orders } (0,0). \quad \text{Degrees } \{3^2 1^3\}. \quad \text{One form.}$$

The dual form is reducible. For if

$$[234] = (\mathbf{012})\,(\mathbf{013})\begin{pmatrix} & 0 & & 1 & \\ 2 & & 4 & & 3\end{pmatrix}$$

then, by what precedes, [234] is skew with respect to each pair of the symbols 2, 3, 4. But

$$0 \equiv (\mathbf{03}\,41)^2\,(04'\,\mathbf{12})^2 \equiv \begin{vmatrix} \cdot & 4'_0 & (\mathbf{120}) \\ 0_3 & 4'_3 & (\mathbf{123}) \\ (041) & -1_4 & 4_1 1_2 \end{vmatrix}^2$$

$$\equiv 2\,.\,0_3(041)4'_0(\mathbf{120})[4'_3 4_1 1_2 - (\mathbf{123})1_4] = T_1 + T_2, \text{ say;}$$

$$T_1 \equiv 2\,.\,0_3 4'_0 4'_3 4_1 1_2 1_0 0_1 4_2 \equiv 2[0_1 0_3 4_1 4_3\,.\,1_0 1_2 4'_0 4'_2 + 0_1 0_3 1_0 1_2 4_1 4'_0(\mathbf{423})]$$

$$\equiv (\mathbf{423})(\mathbf{410})0_1 0_3 1_0 1_2 \equiv (\mathbf{410})[(\mathbf{023})\,1_4 + (\mathbf{420})\,1_3]\,0_1 0_3 1_2$$

$$\equiv (\mathbf{410})(\mathbf{013})\,0_3 0_2 1_2 1_4 + (\mathbf{410})(\mathbf{120})\,0_4 0_3 1_3 1_2 \equiv -[342] - [423] \equiv -2[234],$$

$$T_2 \equiv -(\mathbf{120})(\mathbf{123})\,0_3 1_4 (\mathbf{4\,10\,0}) \equiv -(\mathbf{120})(\mathbf{123})\,0_3 0_4 1_0 1_4$$

$$\equiv -(\mathbf{120})(\mathbf{103})\,0_3 0_4 1_4 1_2 \equiv [324] \equiv -[234],$$

and hence $[234] \equiv 0$.

*The subsystem* $D_5$

56. Consider now the forms $(ijk)\,(ijl)\,\{km\}\{mi\}\{il\}$. By $[g_2, k \text{ of } \{km\}]$, these are skew in $j$, $k$; by $[g_1, l \text{ of } \{il\}]$ they are skew in $j$, $l$. Hence they are skew in any pair of $j$, $k$, $l$. We classify these forms according to the number of factors of type $(abu)$ involved. If no such brackets appear the form is

$$(ijk)\,(ijl)\begin{pmatrix}k & & m & & i & & l\\ & \xi & & \eta & & \zeta & \end{pmatrix},$$

where $\xi$, $\eta$, $\zeta$ are of type $p$ or $x$. Owing to skew symmetry in $j$, $k$, $l$ this is reducible if $\xi = j$, $k$, $l$ or $m$. Hence $\xi = i$ or $x$. Similarly $\zeta = m$ or $x$. And $\eta \neq i$, $m$. By $[g_2, m_\eta]$ we see that the form reduces if $\eta = j$. By skew symmetry, it reduces if $\eta = k$, $l$. Hence $\eta = x$, and since $\xi$, $\eta$, $\zeta$ must all be different, $\xi = i$, $\zeta = m$. The form is now

$$(ijk)\,(ijl)\begin{pmatrix}k & & m & & i & & l\\ & i & & x & & m & \end{pmatrix},$$

and this reduces by the successive transformations $[g_2, m_x]$, $[g_2, k \text{ of } g_1]$, $[g_1, l_m]$. The dual form reduces similarly.

57. If there is one factor of type $(abu)$ the possibilities are

$$\text{(i)}\quad (ijk)\,(ijl)\,(kmu)\begin{pmatrix} m & & i & & l \\ & \xi & & \eta & \end{pmatrix},$$

$$\text{(ii)}\quad (ijk)\,(ijl)\begin{pmatrix} k & & m \\ & \xi & \end{pmatrix}(miu)\begin{pmatrix} i & & l \\ & \eta & \end{pmatrix},$$

$$\text{(iii)}\quad (ijk)\,(ijl)\begin{pmatrix} k & & m & & i \\ & \xi & & \eta & \end{pmatrix}(ilu),$$

where $\xi$, $\eta$ are of type $p$ or $x$. These forms are skew in $j$, $k$, $l$ and clearly reduce if $\xi = \eta$. The reductions which follow apply equally to the dual forms.

For the form (i), by skew symmetry in $j$, $k$, $l$, $\eta = m$ or $x$. If $\eta = x$ the form is symmetric in $k$, $l$ by $[g_3, i_x]$ followed by $[g_3, l_x]$ and $[g_3, m_x]$. Hence we may take $\eta = m$. By $[g_1, m \text{ of } g_3]$ the form is then equivalent to

$$(mjk)\,(ijl)\,(kiu)\begin{pmatrix} m & & i & & l \\ & \xi & & m & \end{pmatrix} + (imk)\,(ijl)\,(kju)\begin{pmatrix} m & & i & & l \\ & \xi & & m & \end{pmatrix}$$

$$\equiv (mjk)\,(iml)\,(kiu)\begin{pmatrix} j & & i & & l \\ & \xi & & m & \end{pmatrix} + (imk)\,(iml)\,(kju)\begin{pmatrix} j & & i & & l \\ & \xi & & m & \end{pmatrix}$$

by $[g_2, m_\xi]$, and each term reduces by $[g_1, l_m]$. So the forms (i) are reducible.

For the form (ii), by skew symmetry, $\xi \neq j, k, l, m$, so that $\xi$ is $i$ or $x$. By $[g_2, m \text{ of } g_3]$, the form is equivalent to $(ijk)\,(iml)\begin{pmatrix} k & & m \\ & \xi & \end{pmatrix}(jiu)\begin{pmatrix} i & & l \\ & \eta & \end{pmatrix}$. If $\xi = x$ this is clearly reducible. If $\xi = i$ then $[g_1, m_\xi]$ transforms the last form to $(imk)\,(iml)\begin{pmatrix} k & & j \\ & \xi & \end{pmatrix}(jiu)\begin{pmatrix} i & & l \\ & \eta & \end{pmatrix}$. Thus the form (ii) is symmetric in $j$, $m$. As it is skew in $j$, $k$ it thus reduces.

For the form (iii), by skew symmetry, $\xi \neq j, k, l, m$. If $\xi = x$ the form is symmetric in $k$, $l$ by $[g_3, k_\xi]$; since it is skew in $j$, $k$ the form is reducible. If $\xi = i$ the form reduces to type (ii) by $[g_1, m_\xi]$ followed by $[g_1, l \text{ of } g_3]$. Hence the form is always reducible.

58. If there is more than one factor of type $(abu)$ the possibilities are

$$\text{(i)}\quad (ijk)\,(ijl)\begin{pmatrix} k & & m \\ & \xi & \end{pmatrix}(miu)\,(ilu), \qquad \text{(ii)}\quad (ijk)\,(ijl)\,(kmu)\begin{pmatrix} m & & i \\ & \xi & \end{pmatrix}(ilu),$$

$$\text{(iii)}\quad (ijk)\,(ijl)\,(kmu)\,(miu)\begin{pmatrix} i & & l \\ & \xi & \end{pmatrix}, \qquad \text{(iv)}\quad (ijk)\,(ijl)\,(kmu)\,(miu)\,(ilu),$$

where $\xi$ is $p$ or $x$. These forms again are skew in $j$, $k$, $l$. The reductions below apply equally to the dual forms.

For the form (i), by skew symmetry, $\xi \neq j, k, l, m$. If $\xi = x$ the form is symmetric in $k$, $l$ by $[g_4, k_\xi]$. But it has been seen to be skew in $k$, $l$. Hence it is reducible. Thus $\xi = i$. In this case $[g_1, m_\xi]$ followed by $[g_1, l \text{ of } g_4]$ shows that the form is unaltered by the permutation $(jlmk)$. Hence it is skew with respect to the permutation $(jl)\,(jlmk) = (lmk)$. Since this is of period three the form is reducible.

For the form (ii), $\xi \neq i, m$. By $[g_2, m_\xi]$ it is reducible if $\xi = j$. By skew symmetry it is also reducible if $\xi = k, l$. The only other possibility is $\xi = x$. In this case $[g_3, i_\xi]$ reduces (ii) to a form of type (i) already considered.

The form (iii) reduces to type (ii) by $[g_2, m \text{ of } g_4]$ followed by $[g_1, l_\xi]$, and is accordingly reducible.

The form (iv) is symmetric in $k$, $l$ by $[g_3, l \text{ of } g_5]$. But it has been seen to be skew in $k$, $l$. Hence it reduces.

*The subsystem* $D_6$

59. The subsystem $D_6$ consists of forms $(ijk)\,(ilm)\,\{jl\}\{km\}$. If neither of the last two factors is of the form $(abu)$ the form is $(ijk)\,(ilm)\begin{pmatrix} j & l \\ & \xi \end{pmatrix}\begin{pmatrix} k & m \\ & \eta \end{pmatrix}$, where $\xi$, $\eta$ are of type $p$ or $x$. All these forms can be expressed in terms of reducible forms and forms of the system $D_4$ [without factors $(abu)$] already considered. For, all equivalences being *modulo* forms $D_4$, the form is skew in $i$, $l$ by $[g_1, l_\xi]$, skew in $i, j$ by $[g_2, j_\xi]$, skew in $i$, $k$ by $[g_2, k_\eta]$ and skew in $i$, $m$ by $[g_1, m_\eta]$. Hence it is skew in any pair of $ijklm$. Hence it is certainly reducible *modulo* forms $D_4$ if $\xi$, $\eta$ take any values in the set $ijklm$. Thus the only possibility remaining is $\xi = \eta = x$. In this case the form is $(ijk)\,(ilm)\,j_x k_x l_x m_x$ and is an alternating function of $ijklm$, so that it is unaltered by cyclic permutation of $jkl$. But, by $[g_1, l \text{ of } g_2]$, the sum of the three forms obtained under this cyclic permutation is zero. Thus the form is reducible to forms $D_4$ in each case. The same reasoning applies to the dual forms.

We may thus confine our attention to forms containing at least one factor $(abu)$. The possibilities are

(i) $(ijk)\,(ilm)\,(jlu)\,(kmu)$, (ii) $(ijk)\,(ilm)\,(jlu)\begin{pmatrix} k & m \\ & p \end{pmatrix}$, (iii) $(ijk)\,(ilm)\,(jlu)\,k_x m_x$.

These prove to give rise to irreducible forms, but the relations between them are, as we shall see, rather involved. The three cases will be treated in order.

60. We write
$$[ijklm] = (ijk)\,(ilm)\,(jlu)\,(kmu), \tag{60·1}$$
$ijklm$ being a permutation of 01234. Clearly
$$[ijklm] \equiv [ikjml] \equiv [ilmjk] \equiv [imlkj]. \tag{60·2}$$
Thus there are 30 forms to consider, 6 corresponding to each value of the initial symbol $i$.

We consider the six term sum
$$\sum_{ijk} [ijklm] \equiv (ijk) \dot{\sum} (ilm)\,(jlu)\,(kmu) \equiv (ijk)^2\,(lm\;lu\;mu),$$
the dot indicating a determinantal permutation of $ijk$. Thus
$$\sum_{ijk} [ijklm] \equiv 0. \tag{60·3}$$
Again,
$$\begin{aligned} [ijklm] + [mjkli] &\equiv (ilm)\,(jlu)\,[(ijk)\,(kmu) - (mjk)\,(kiu)] \\ &\equiv (ilm)\,(jlu)\,(imk)\,(kju) \\ &\equiv [ijlkm] + [mjlki] \end{aligned}$$
since the expression is symmetric in $k$, $l$. Thus
$$[ijklm] - [ijlkm] \equiv -([mjkli] - [mjlki]). \tag{60·4}$$

And $[ijklm]+[ijmlk] \equiv (jlu)(kmu)[(ijk)(ilm)-(ijm)(ilk)] \equiv (jlu)(kmu)(imk)(ilj)$
$$\equiv (imk)(ilj)[(mlu)(kju)+(jmu)(klu)] \equiv [ijlkm]+[ijlmk],$$
so that
$$[ijklm]-[ijlkm] \equiv -([ijmlk]-[ijlmk]). \qquad (60{\cdot}5)$$

Let $\lambda$ denote the left-hand side of (60·4) and (60·5). Then $\lambda$ is skew in each of the pairs of symbols $kl$, $im$, $km$; it is symmetric for the permutation $(jm)(kl)$, by (60·1), and hence skew in $(jm)$. Thus $\lambda$ is an alternating function of the symbols 01234, and it takes two values, differing only in sign, according as $ijklm$ is an odd or even permutation of 01234. Evidently all the forms can be expressed in terms of those corresponding to even permutations of 01234, together with $\lambda$. To fix the sign we suppose that $\lambda$ is the value of (60·4) for $ijklm$ an even permutation.

By (60·2), the forms to be considered may be taken to be

$$\left.\begin{aligned} a_i &= [i,i+1,i+2,i+3,i+4], & a_i' &= [i,i+1,i+3,i+2,i+4],\\ b_i &= [i,i+1,i+3,i+4,i+2], & b_i' &= [i,i+1,i+4,i+3,i+2],\\ c_i &= [i,i+1,i+4,i+2,i+3], & c_i' &= [i,i+1,i+2,i+4,i+3],\end{aligned}\right\} \qquad (60{\cdot}6)$$

where $i = 0, 1, 2, 3, 4$ and the suffixes are reduced *modulo* 5. We then have

$$a_i - a_i' \equiv b_i - b_i' \equiv c_i - c_i' \equiv \lambda. \qquad (60{\cdot}7)$$

We have now to consider the relations between these forms given by (60·3). There is one such relation for each ordered pair $l$, $m$; but from (60·2) it is immediately seen that the relation with $m$, $l$ interchanged is equivalent to the original one. There are thus ten of these relations, five corresponding to values $i+3$, $i+4$ of $l$, $m$ which are consecutive, and five corresponding to values $i+2$, $i+4$ which are non-consecutive. Using (60·2) and (60·7) it is easily seen that these relations can be written

$$a_i+a_{i+2}+b_{i+1}+c_i+c_{i+1}+c_{i+2} \equiv 3\lambda, \quad a_i+a_{i+1}+b_i+b_{i+1}+b_{i+3}+c_{i+3} \equiv 3\lambda \qquad (60{\cdot}8)$$

for $i = 0, 1, 2, 3, 4$. These relations lead to

$$a_i \equiv \tfrac{3}{2}\lambda - b_{i+1} - b_{i-1}, \quad c_i \equiv b_{i+1}+b_{i-1}-b_i \qquad (60{\cdot}9)$$

and show that all the forms are expressible in terms of six independent ones, e.g. the forms $b_i$ and $\lambda$. So we have the irreducible set

[75] (012) (034) (13$u$) (24$u$) Orders (0, 2). Degrees $\{1^5\}$. Six forms.

61. The duals of these forms are reducible. We use the corresponding notation to denote these forms. In the first place

$$[01234] \equiv (\mathbf{012})(\mathbf{034})(\mathbf{13}x)(\mathbf{24}x) \equiv 2\,.\,0_1 0_2'(0_3 0_4' - 0_4 0_3')(\mathbf{13}x)(\mathbf{24}x)$$
$$\equiv -2\,.\,0_1 0_4 0_2' 0_3' (\mathbf{13}x)(\mathbf{24}x).$$

By $[g_1, \mathbf{2}$ of $g_2]$ this is symmetric in **2** and **3**. Hence $\lambda = 0$, and (60·9) gives

$$a_i \equiv -b_{i+1}-b_{i-1}, \quad c_i \equiv b_{i+1}+b_{i-1}-b_i, \qquad (61{\cdot}1)$$

while (60·7) gives
$$a_i' \equiv a_i, \quad b_i' \equiv b_i, \quad c_i' \equiv c_i. \qquad (61{\cdot}2)$$

Next,

$$0 \equiv (012)\,0_3\,1_3\,2_x\,.\,(0'1'2')\,0'_4\,1'_x\,2'_4 \equiv [(012)\,(01'2')\,0'_3\,0'_4 + (012)\,0'_4\,(\mathbf{03}\,1'2')]\,1_3\,2_x\,1'_x\,2'_4$$
$$\equiv [(012)\,(012')\,1'_3\,1'_x + (012)\,1'_x(\mathbf{13}\,2'0)]\,0'_3\,0'_4\,2_x\,2'_4 + (012)\,0'_4(\mathbf{03}\,1'2')\,1_3\,2_x\,1'_x\,2'_4$$
$$= T_1 + T_2 + T_3, \text{ say,}$$

$$T_1 \equiv -\tfrac{1}{2}(\mathbf{2}\,01\,x)\,(\mathbf{2}\,01\,\mathbf{4})\,1'_3\,1'_x\,0'_3\,0'_4 \equiv \tfrac{1}{2}0_2\,1_2[0_x\,1_4 + 0_4\,1_x]\,1'_3\,1'_x\,0'_3\,0'_4 = T_{11} + T_{12}, \text{ say,}$$

$$T_{11} \equiv \tfrac{1}{2}[0_2\,0_x\,1_2\,1_x\,.\,0'_3\,0'_4\,1'_3\,1'_4 + 0_2\,0_x\,0'_3\,0'_4\,1_2\,1'_3(\mathbf{14}x)] \equiv \tfrac{1}{4}(\mathbf{123})\,(\mathbf{14}x)\,0_2\,0_x\,0'_3\,0'_4$$
$$\equiv \tfrac{1}{4}[(\mathbf{23}x)\,0_1 + (\mathbf{12}x)\,0_3]\,(\mathbf{14}x)\,0_2\,0'_3\,0'_4$$
$$\equiv \tfrac{1}{4}(\mathbf{23}x)\,(\mathbf{14}x)\,[0_1\,0_4\,0'_2\,0'_3 + 0_1\,0'_3(\mathbf{024})] - \tfrac{1}{8}(\mathbf{12}x)\,(\mathbf{14}x)\,(\mathbf{032})\,(\mathbf{034})$$
$$\equiv \tfrac{1}{8}[(\mathbf{23}x)\,(\mathbf{14}x)\,(\mathbf{013})\,(\mathbf{024}) - (\mathbf{032})\,(\mathbf{12}x)\,\{(\mathbf{04}x)\,(\mathbf{134}) + (\mathbf{34}x)\,(\mathbf{014})\}]$$
$$\equiv \tfrac{1}{8}([01342] - [34102] - [01423]) \equiv \tfrac{1}{8}(b_0 - a'_3 - c_0),$$

$$T_{12} \equiv -\tfrac{1}{4}(\mathbf{1}x\mathbf{2})\,(\mathbf{1}x\mathbf{3})\,0_2\,0_4\,0'_3\,0'_4 \equiv \tfrac{1}{8}(\mathbf{042})\,(\mathbf{043})\,(\mathbf{21}x)\,(\mathbf{31}x)$$
$$\equiv \tfrac{1}{8}(\mathbf{042})\,(\mathbf{21}x)\,[(\mathbf{143})\,(\mathbf{30}x) + (\mathbf{013})\,(\mathbf{34}x)]$$
$$\equiv \tfrac{1}{8}([40231] + [01324]) \equiv \tfrac{1}{8}(b_4 + a'_0),$$

$$T_2 \equiv \tfrac{1}{2}(\mathbf{1}\,20\,x)\,(\mathbf{13}\,2'0)\,0'_3\,0'_4\,2_x\,2'_4 \equiv -\tfrac{1}{2}2_1\,0_x[2'_1\,0_3 - 2'_3\,0_1]\,0'_3\,0'_4\,2_x\,2'_4.$$

The second term is reducible, and hence

$$T_2 \equiv -\tfrac{1}{8}(\mathbf{21}x)\,(\mathbf{214})\,(\mathbf{03}x)\,(\mathbf{034}) \equiv -\tfrac{1}{8}[(\mathbf{20}x)\,(\mathbf{13}x) + (\mathbf{23}x)\,(\mathbf{01}x)]\,(\mathbf{214})\,(\mathbf{034})$$
$$\equiv \tfrac{1}{8}(-[40321] - [40312]) \equiv \tfrac{1}{8}(-b'_4 - c_4).$$

$$T_3 \equiv \tfrac{1}{2}(\mathbf{0}\,12\,\mathbf{4})\,(\mathbf{03}\,1'2')\,1_3\,2_x\,1'_x\,2'_4 \equiv \tfrac{1}{2}[1_4\,2_0 - 1_0\,2_4]\,[1'_0\,2'_3 - 1'_3\,2'_0]\,1_3\,2_x\,1'_x\,2'_4$$
$$\equiv \tfrac{1}{2}[-1_4\,2_0\,1'_3\,2'_0\,1_3\,2_x\,1'_x\,2'_4 - 1_0\,2_4\,1'_0\,2'_3\,1_3\,2_x\,1'_x\,2'_4 + 1_0\,2_4\,1'_3\,2'_0\,1_3\,2_x\,1'_x\,2'_4]$$
$$\equiv \tfrac{1}{8}[-(\mathbf{134})\,(\mathbf{13}x)\,(\mathbf{204})\,(\mathbf{20}x) - (\mathbf{103})\,(\mathbf{10}x)\,(\mathbf{243})\,(\mathbf{24}x) + (\mathbf{130})\,(\mathbf{13}x)\,(\mathbf{240})\,(\mathbf{24}x)]$$
$$\equiv \tfrac{1}{8}[-(\mathbf{134})\,(\mathbf{204})\,\{(\mathbf{23}x)\,(\mathbf{10}x) + (\mathbf{12}x)\,(\mathbf{30}x)\} - (\mathbf{103})\,(\mathbf{243})\,\{(\mathbf{20}x)\,(\mathbf{14}x) + (\mathbf{12}x)\,(\mathbf{04}x)\}$$
$$+ (\mathbf{130})\,(\mathbf{240})\,\{(\mathbf{23}x)\,(\mathbf{14}x) + (\mathbf{12}x)\,(\mathbf{34}x)\}]$$
$$\equiv \tfrac{1}{8}(-[40213] - [40231] - [34210] - [34201] + [01342] + [01324])$$
$$\equiv \tfrac{1}{8}(-a'_4 - b_4 - b'_3 - c_3 + b_0 + a'_0).$$

Hence $\quad (b_0 - a'_3 - c_0) + (b_4 + a'_0) - (b'_4 + c_4) - (a'_4 + b_4 + b'_3 + c_3 - b_0 - a'_0) \equiv 0.$

Using (61·1) and (61·2) this gives $b_0 - b_1 - b_4 \equiv 0$ or $c_0 \equiv 0$. Similarly each $c_i \equiv 0$. Whence, from (61·1), $a_i \equiv b_i \equiv 0$ and all the forms are reducible.

62. Next consider the forms (ii) of § 59. Here *ijklm* are 01234 in some order, and we may suppose, for definiteness, that $p = 0$. Then the form is reducible if $k, m = 0$. Since the form is skew with respect to the permutation $(jl)\,(km)$ we have to consider the two types

$$[jklm] \equiv (0jk)\,(0lm)\,(jlu)\begin{pmatrix} k & m \\ & 0 \end{pmatrix}, \qquad [jklm]' \equiv (j0k)\,(jlm)\,(0lu)\begin{pmatrix} k & m \\ & 0 \end{pmatrix}, \qquad (62·1)$$

where *jklm* is a permutation of 1234. Now

$$[jklm] + [jklm]' \equiv (0jk)\begin{pmatrix} k & m \\ & 0 \end{pmatrix}[(0lm)\,(jlu) - (jlm)\,(0lu)]$$
$$\equiv (0jk)\begin{pmatrix} k & m \\ & 0 \end{pmatrix}(0lj)\,(mlu) \equiv [mklj], \quad \text{by } [g_1, m_0],$$

and so $\qquad [jklm]' \equiv [mklj] - [jklm], \qquad (62·2)$

and the second set of forms is expressible in terms of the first. By $[g_1, m_0]$, $[g_2, k_0]$, $[g_1, j_0]$ applied in succession we see that $[jklm]$ is skew in $k$ and $m$. Since it is skew with respect to the permutation $(jl)(km)$ it is symmetric in $j$ and $l$. Accordingly the notation may be simplified by writing

$$[jklm] = [km] \equiv -[mk]. \tag{62·3}$$

By applying $[g_2, k_0]$ to $[jklm]'$ we find

$$[jklm]' + [kjlm]' \equiv (j0k)(jkm)(0lu)\begin{pmatrix} l & & m \\ & 0 & \end{pmatrix}$$

$$\equiv [kljm]' + [jlkm]', \quad \text{by } [g_3, j \text{ of } g_1].$$

Hence, from (62·2) and (62·3),

$$[kj] - [km] + [jk] - [jm] \equiv [lk] - [lm] + [lj] - [lm],$$

which, since $[ab] \equiv -[ba]$, can be written

$$[mk] + [kl] + [lm] \equiv [jm] + [ml] + [lj]$$
$$\equiv [jk] + [km] + [mj] \equiv [jl] + [lk] + [kj], \tag{62·4}$$

by cyclic permutation of $klm$, which leaves the left-hand member of (62·4) unchanged. By addition we see that each expression is equivalent to zero.

Thus

$$[mk] + [kl] + [lm] \equiv 0, \tag{62·5}$$

from which it follows that all the forms $[ij]$ are expressible in terms of three, for instance, [12], [13], [24]. So we get the irreducible set

[76] $(012)(034)(13u)\begin{pmatrix} 2 & & 4 \\ & 0 & \end{pmatrix}$ Orders (0, 1). Degrees $\{31^4\}$. Three forms.

63. The dual forms are reducible. For, using an analogous notation,

$$0 \equiv (\mathbf{02}x)\, 1_0 1_2 . 0_3 0_4 1_3' 1_4' \equiv [(\mathbf{32}x)\, 1_0' + (\mathbf{03}x)\, 1_2' + (\mathbf{023})\, 1_x']\, 0_3 0_4 1_0 1_2 1_4'$$
$$= T_1 + T_2 + T_3, \text{ say},$$

$$T_1 \equiv -\tfrac{1}{2}(\mathbf{102})(\mathbf{104})(\mathbf{32}x)\begin{pmatrix} & 0 & \\ 3 & & 4 \end{pmatrix} \equiv -\tfrac{1}{2}(\mathbf{102})(\mathbf{304})(\mathbf{32}x)\begin{pmatrix} & 0 & \\ 1 & & 4 \end{pmatrix}$$
$$\equiv -\tfrac{1}{2}[2134] \equiv -\tfrac{1}{2}[14],$$

$$T_2 \equiv -\tfrac{1}{2}(\mathbf{120})(\mathbf{124})(\mathbf{03}x)\begin{pmatrix} & 0 & \\ 3 & & 4 \end{pmatrix} \equiv -\tfrac{1}{2}(\mathbf{124})[(\mathbf{320})(\mathbf{01}x) + (\mathbf{130})(\mathbf{02}x)]\begin{pmatrix} & 0 & \\ 3 & & 4 \end{pmatrix}$$
$$\equiv \tfrac{1}{2}([2314]' + [1324]') \equiv \tfrac{1}{2}([32] - [34] + [31] - [34]),$$

$$T_3 \equiv (\mathbf{043})\, 1_x' 0_3 0_2 1_0 1_2 1_4' \equiv [(\mathbf{43}x)\, 1_0' + (\mathbf{04}x)\, 1_3']\, 0_3 0_2 1_0 1_2 1_4' = T_{31} + T_{32}, \text{ say},$$

$$T_{31} \equiv -\tfrac{1}{2}(\mathbf{102})(\mathbf{104})(\mathbf{43}x)\begin{pmatrix} & 0 & \\ 2 & & 3 \end{pmatrix} \equiv -\tfrac{1}{2}(\mathbf{302})(\mathbf{104})(\mathbf{43}x)\begin{pmatrix} & 0 & \\ 2 & & 1 \end{pmatrix} \equiv \tfrac{1}{2}[4132] \equiv \tfrac{1}{2}[12],$$

$$T_{32} \equiv (\mathbf{04}x)\, 1_0 1_4 . 0_3 0_2 1_3' 1_2' + (\mathbf{04}x)\, 1_0 0_3 0_2 1_3' (\mathbf{124}) \equiv \tfrac{1}{2}(\mathbf{103})(\mathbf{124})(\mathbf{04}x)\begin{pmatrix} & 0 & \\ 2 & & 3 \end{pmatrix}$$
$$\equiv -\tfrac{1}{2}[1342]' \equiv \tfrac{1}{2}([32] - [31]).$$

Hence

$$0 \equiv -[14] + [32] - [34] + [31] - [34] + [12] + [32] - [31]$$
$$\equiv 3 . [42],$$

using (62·5) to reduce the expression on the right. Hence $[42] \equiv 0$, and the other forms reduce in the same way.

64. The last form of this set is

$$[ijklm] \equiv (ijk)\,(ilm)\,(jlu)\,k_x m_x. \tag{64·1}$$

Clearly

$$[ijklm] \equiv -[ilmjk]. \tag{64·2}$$

By $[g_1, m_x]$,

$$[ijklm] \equiv -[mjkli] + (imk)\,(ilm)\,(jlu)\,k_x j_x. \tag{64·3}$$

By $[g_3, k_x]$ the second form on the right is symmetrical in $k$ and $l$. Whence

$$[ijklm] - [ijlkm] \equiv -([mjkli] - [mjlki]). \tag{64·4}$$

From (64·3), by $[g_2, j \text{ of } g_3]$ applied to the second term on the right,

$$[ijklm] + [mjkli] \equiv [miklj] + [imklj].$$

By (64·2), this can be written

$$-[ilmjk] - [mlijk] \equiv -[mljik] - [iljmk]$$

or

$$[iljmk] - [ilmjk] \equiv -([mljik] - [mlijk]).$$

Replacing $l, j, m, k$ by $j, k, l, m$ respectively, this becomes

$$[ijklm] - [ijlkm] \equiv -([ljkim] - [ljikm]). \tag{64·5}$$

Again, by $[g_2, j \text{ of } g_1]$,

$$[ijklm] \equiv [ijmlk] + (imk)\,(ilj)\,(jlu)\,k_x m_x$$
$$\equiv [ijmlk] + [iljkm] + [ijlkm],$$

by $[g_3, k_x]$ applied to the second term on the right. Hence, using (64·2),

$$[ijklm] - [ijlkm] \equiv -([ilkjm] - [iljkm]). \tag{64·6}$$

The expression common to the left-hand members of (64·4), (64·5) and (64·6) is clearly skew in $k, l$. By (64·4), (64·5) and (64·6) it is skew in each of the pairs $i, m$; $i, l$; $j, l$. Hence it is an alternating function of $ijklm$, taking two values (differing in sign) according as $ijklm$ is an even or an odd permutation of 01234. Let $\lambda$ denote the value of this expression when $ijklm$ is 01234.

Put, now, $\xi = [01234]$, and consider the eight forms obtained by permuting symbols according to the group generated by (23) and (13) (24). Using (64·2) and the property of $\lambda$ just established, it is easily seen that

$$[01234] \equiv \xi, \quad [01324] \equiv \xi - \lambda, \quad [03142] \equiv -\xi - \lambda, \quad [02143] \equiv -\xi + 2\lambda,$$
$$[03412] \equiv -\xi, \quad [02413] \equiv -\xi + \lambda, \quad [04231] \equiv \xi + \lambda, \quad [04321] \equiv \xi - 2\lambda.$$

But $[04321] - [04231] \equiv \lambda$, and hence $\lambda \equiv 0$.

Accordingly, $[ijklm]$ is symmetrical in $k, l$, skew with respect to the permutation $(jl)\,(km)$, and hence symmetric in $j, m$, since

$$(jl)\,(km)\,.\,(kl)\,.\,(jl)\,(km) = (jm).$$

Thus we may write, more briefly,

$$[ijklm] \equiv [ijm] \equiv [imj] \equiv -[ikl] \equiv -[ilk]. \tag{64·7}$$

In view of the symmetry relations established, there are only 15 forms to consider, namely

$$\left.\begin{aligned} a_i &\equiv [i, i+1, i+2] \equiv -[i, i+3, i+4],\\ b_i &\equiv [i, i+1, i+3] \equiv -[i, i+2, i+4],\\ c_i &\equiv [i, i+1, i+4] \equiv -[i, i+2, i+3], \end{aligned}\right\} \tag{64·8}$$

where $i = 0, 1, 2, 3, 4$ and the expressions $i+p$ are reduced *modulo* 5.

Now, permuting $ijk$ in all six possible ways and adding,

$$\sum_{ijk} [ijklm] \equiv (ijk) \dot{\sum} (ilm)(jlu)\, k_x m_x \equiv (ijk)^2 (lm\, lu\, x)\, m_x,$$

and so

$$\sum_{ijk} [ijklm] \equiv 0. \tag{64·9}$$

Since $[ijklm] \equiv -[ilmjk] \equiv -[ikjml]$, (64·9) leads to ten relations between the fifteen forms (64·8), corresponding to the ten pairs $l$, $m$ in the set 01234. Putting $l, m = i+3$, $i+4$ and $i+2$, $i+4$ in (64·9), we obtain

$$-a_{i+1} + (-b_i + b_{i+1} - b_{i+2}) + (c_i - c_{i+2}) \equiv 0,$$
$$(-a_i - a_{i+1} + a_{i+3}) + b_{i+3} + (c_i - c_{i+1}) \equiv 0.$$

These relations are seen, after a little reduction, to imply

$$a_{i+1} + b_i + b_{i+2} \equiv 0, \quad b_{i+1} + c_i - c_{i+2} \equiv 0, \tag{64·10}$$

so that all the forms are expressible in terms of $c_0, c_1, \ldots, c_4$. This gives the irreducible set

[77] $(012)(034)(13u)\, 2_x 4_x$ Orders $(2, 1)$. Degrees $\{1^5\}$. Five forms.

65. The dual forms are reducible. For, using the corresponding notation,

$$\begin{aligned} 0 &\equiv (\mathbf{13}x)\, 0_1 0_3 \,.\, 0'_2 0'_4 u_2 u_4 \equiv [(\mathbf{23}x)\, 0'_1 + (\mathbf{12}x)\, 0'_3]\, 0_1 0_3 0'_4 u_2 u_4 \\ &\equiv -\tfrac{1}{2}[(\mathbf{23}x)(\mathbf{013})(\mathbf{014}) + (\mathbf{12}x)(\mathbf{031})(\mathbf{034})]\, u_2 u_4 \\ &\equiv -\tfrac{1}{2}[(\mathbf{013})(\mathbf{23}x)\{(\mathbf{214})\, u_0 + (\mathbf{024})\, u_1\} u_4 + (\mathbf{031})(\mathbf{12}x)\{(\mathbf{234})\, u_0 + (\mathbf{024})\, u_3\} u_4] \\ &\equiv \tfrac{1}{2}([12430] + [02431] + [31024] + [01324]) \equiv \tfrac{1}{2}([120] + [021] + [314] + [014]) \\ &\equiv \tfrac{1}{2}(c_1 + a_0 + b_3 + c_0) \equiv \tfrac{1}{2}(c_0 + c_1 - 2c_2 + c_3 + c_4), \quad \text{by (64·10)}. \end{aligned}$$

Thus $3c_2 = (c_0 + c_1 + c_2 + c_3 + c_4)$. Hence, by adding similar expressions, $c_i \equiv 0$.

## *The system* E

66. This system consists of forms $PQ$, where $P$ is a product of three brackets of type $(abc)$ and $Q$ contains no such brackets, and the duals of these forms. As in the case of forms C and D, the first class may be taken to be representative, and $Q$ consists of generalized chains and tags. Four cases can be distinguished according to the type of the product $P$, giving rise to four subsystems $E_1, \ldots, E_4$, namely,

$$E_1 \quad (ijk)(ijl)(ijm)\, Q, \qquad E_2 \quad (ijk)(ijl)(ikm)\, Q,$$
$$E_3 \quad (ijk)(ijl)(ikl)\, Q, \qquad E_4 \quad (ijk)(ijl)(klm)\, Q.$$

These subsystems will now be considered in turn.

67. In the form $(ijk)(ijl)(ijm)\,Q$ of $E_1$ each symbol appears an odd number of times in $Q$. If $Q$ contains a generalized chain $\{k,l\}$, $\{k,m\}$ or $\{l,m\}$ the form is clearly reducible. So the only type to consider is typified by the form $(ijk)(ijl)(ijm)\{i,k\}\{j,l\}m_x$, and this reduces by $[g_2,\ k$ of $\{i,k\}]$. Since this argument applies to the dual forms equally, no irreducible forms arise from the subsystem $E_1$.

68. Consider next the form $(ijk)(ijl)(ikm)\,Q$ of $E_2$. Here $Q$ must contain $i$, $l$, $m$ an odd number of times. If $Q$ contains a generalized chain $\{i,j\}$ the form reduces to forms of $E_1$ by $[g_3, j$ of $\{i,j\}]$. Similarly it reduces if it contains $\{i,k\}$. It is clearly reducible if $Q$ contains $\{k,l\}$ or $\{j,m\}$, and is reducible by $[g_2,\ m$ of $g_3]$ if $Q$ contains $\{k,m\}$ or $\{j,l\}$. Thus neither $j$ nor $k$ can occur in $Q$ if the form is irreducible. If $Q$ is $\{il\}\{lm\}l_x$ the form reduces by $[g_1,\ l$ of $\{il\}]$. And if $Q$ is $\{li\}\{im\}i_x$ the form reduces, for it is changed in sign after application of the six transformations $[g_1,\ l$ of $\{li\}]$, $[g_3,\ j$ of $\{ji\}]$, $[g_2,\ k$ of $\{ki\}]$, $[g_3,\ l$ of $\{li\}]$, $[g_1,\ j$ of $\{ji\}]$, $[g_3,\ k$ of $\{ki\}]$. Thus, neglecting varieties produced by a permutation of symbols, the only possible forms for $Q$ are $\{il\}m_x$ and $\{lm\}i_x$, and the following possibilities arise:

(i) $(ijk)(ijl)(ikm)(ilu)\,m_x$, (ii) $(ijk)(ijl)(ikm)\begin{pmatrix} i & & l \\ & p & \end{pmatrix} m_x$,

(iii) $(ijk)(ijl)(ikm)\,i_x l_x m_x$, (iv) $(ijk)(ijl)(ikm)(lmu)\,i_x$,

(v) $(ijk)(ijl)(ikm)\begin{pmatrix} l & & m \\ & p & \end{pmatrix} i_x$.

The form (i) is skew in $j$, $l$ by $[g_1,\ l$ of $g_4]$, and skew in $j$, $k$ by $[g_2,\ k$ of $g_3]$. Thus it is skew in each pair of $j$, $k$, $l$ and gives rise to four forms of an irreducible set, namely

[78] $(012)(013)(024)(03u)\,4_x$ Orders $(1,1)$. Degrees $\{21^4\}$. Four forms.

The dual form is reducible, since

$$(\mathbf{012})(\mathbf{013})(\mathbf{024})(\mathbf{03}x)\,u_4 \equiv -2[1_0 1'_0 1_2 1'_3(\mathbf{024})(\mathbf{03}x)\,u_4] \equiv 0.$$

The form (ii) is reducible if $p = i, l$. If $p = j$ it reduces by $[g_1, l_p]$. If $p = k$ it reduces by $[g_2,\ k$ of $g_3]$ followed by $[g_1, l_p]$. Finally, if $p = m$, the form is skew in $j$, $k$ by $[g_2,\ k$ of $g_3]$ and symmetric in $k$, $l$ by $[g_3, l_p]$. Hence it reduces. The same reductions apply to the dual form.

The form (iii) and its dual are reducible. For (iii) is symmetric in $k$, $l$ by $[g_3, l_x]$ and skew in $j$, $l$ by $[g_1, l_x]$, and hence reduces.

The form (iv) is expressible in terms of forms (i) by $[g_4, i_x]$.

The form (v) is reducible if $p = l, m$. It is symmetric in $l$, $m$ by $[g_2,\ m$ of $g_3]$. If $p = i$ the form is skew in $j$, $l$ by $[g_1, l_p]$. Hence, being symmetric in $l$, $m$ it is reducible. Suppose that $p = j$. By $[g_1, l_p]$ the form is equivalent to $(ljk)(ijl)(ikm)\begin{pmatrix} i & & m \\ & p & \end{pmatrix} i_x$, and by $[g_1, i_x]$ this is expressible in terms of forms of type (ii). Similarly the form reduces to type (ii) if $p = k$, using $[g_1, m_p]$ and $[g_1,\ i_x]$. Since these reductions apply equally to the dual forms, no new forms arise.

69. Consider next the form $(ijk)(ijl)(ikl)\,Q$ of $E_3$. The symbol $i$ occurs an odd number of times in $Q$. If $Q$ contains a factor $\{i,j\}$ the form reduces by $[g_3, j$ of $\{i,j\}]$. Similarly it reduces if $Q$ contains $\{i,k\}$ or $\{i,l\}$. If $Q$ contains $\{i,m\}$ the form reduces to type $E_2$ by $[g_3,\ m$ of $\{i,m\}]$.

Hence $Q$ can only be $i_x$ (Turnbull 1910). The form then belongs to four quadratics and gives the irreducible set

[79] (012) (013) (023) $0_x$ Orders (1, 0). Degrees $\{21^3\}$. One form.

The dual form is reducible (Todd 1948).

70. The forms $E_4$ are $(ijk)\,(ijl)\,(klm)\,Q$. Here $Q$ must be of the form $\{m, x\}$. If $i$ or $j$ occurs in this generalized tag it can be brought into $g_3$ and the form expressed in terms of forms $E_2$ and $E_3$. If $k$ occurs in $Q$ it can be brought into $g_2$, and the form reduced to type $E_2$. Similarly, the form reduces if $l$ occurs in $Q$. Hence the only type of form to consider is

$$[klm] \equiv (ijk)\,(ijl)\,(klm)\,m_x, \tag{70·1}$$

the notation being justified since the form is clearly symmetric in $i, j$. Since it is skew in $k$, $l$

$$[klm] \equiv -[lkm]. \tag{70·2}$$

By $[g_2, k \text{ of } g_3]$,

$$[ilm]+[jlm]+[klm] \equiv 0. \tag{70·3}$$

And

$$\begin{aligned}[klm]+[mlk] &\equiv (ijl)\,(klm)\,[(ijk)\,m_x-(ijm)\,k_x] \\ &\equiv (ijl)\,(klm)\,[(mjk)\,i_x+(imk)\,j_x] \\ &\equiv -[jli]-[ilj],\end{aligned}$$

so that

$$[klm]+[mlk]+[ilj]+[jli] \equiv 0. \tag{70·4}$$

It is easily verified that these relations enable all the forms of this type to be expressed in terms of ten suitably chosen ones, for instance the ten forms $[i, i+1, i+4]$, $[i, i+3, i+2]$ for $i = 0, 1, 2, 3, 4$, the expressions $i+p$ being reduced *modulo* 5. Hence we have the irreducible set

[80] (012) (013) (234) $4_x$ Orders (1, 0). Degrees $\{1^5\}$. Ten forms.

The dual forms are reducible. For

$$0 \equiv (23u)\,2_0 3_0 . 2'_1 3'_1 2'_4 3'_4 \equiv [(2'3u)\,2_1+(22'u)\,3_1+(232')\,u_1]\,2_0 3_0 3'_1 2'_4 3'_4 = T_1+T_2+T_3, \text{ say,}$$

$$\begin{aligned}T_1 &\equiv (2'3'u)\,2'_4 3'_4 . 2_1 2_0 3_1 3_0 + 2'_4 3'_4 2_1 2_0 3_0 (\mathbf{3}\,u 2'\,\mathbf{1}) \equiv \tfrac{1}{2}(\mathbf{304})\,[u_1 2'_3 - u_3 2'_1]\,2_0 2_1 2'_4 \\ &\equiv \tfrac{1}{2}[\{(\mathbf{104})\,2_3+(\mathbf{301})\,2_4\}\,2_0 2'_4 2'_3 u_1 - (\mathbf{304})\,2_0 2_1 2'_1 2'_4 u_3] \\ &\equiv -\tfrac{1}{4}[(\mathbf{104})\,(\mathbf{230})\,(\mathbf{234})\,u_1+(\mathbf{301})\,(\mathbf{240})\,(\mathbf{243})\,u_1-(\mathbf{304})\,(\mathbf{210})\,(\mathbf{214})\,u_3] \\ &\equiv \tfrac{1}{4}(-[041]-[301]+[043]),\end{aligned}$$

$$T_2 \equiv -\tfrac{1}{4}(\mathbf{310})\,(\mathbf{314})\,(\mathbf{204})\,u_2 \equiv \tfrac{1}{4}(-[042]),$$

$$\begin{aligned}T_3 &\equiv -\tfrac{1}{2}(\mathbf{204})\,3_0 3_2 3'_1 3'_4 u_1 \equiv -\tfrac{1}{2}[(\mathbf{104})\,3'_2+(\mathbf{214})\,3'_0]\,3_0 3_2 3'_4 u_1 \\ &\equiv \tfrac{1}{4}[(\mathbf{104})\,(\mathbf{320})\,(\mathbf{324})\,u_1+(\mathbf{214})\,(\mathbf{302})\,(\mathbf{304})\,u_1] \equiv \tfrac{1}{4}([041]+[421]).\end{aligned}$$

Hence

$$-[301]+[043]-[042]+[421] \equiv 0.$$

Interchange 2, 4 and add. Since, by (70·2), (70·3) and (70·4),

$$\begin{aligned}[043]+[023] &\equiv -[403]-[203] \equiv [103], \\ [042]+[024] &\equiv -[402]-[204], \\ [421]+[241] &\equiv 0, \\ [103]+[402]+[204] &\equiv -[301],\end{aligned}$$

it follows that $[301] \equiv 0$. Similarly the other forms reduce.

## Completeness of the system

71. The forms of the sets [1] to [80] include all the irreducible concomitants of five or fewer quadratics which contain not more than three factors of type $(abc)$ or $(\mathbf{pqr})$. But any form containing four such factors is reducible (Turnbull 1910). For consider such a form, $g_1 g_2 g_3 g_4 Q$ say, where $g_i$ are of the type $(abc)$ or $(\mathbf{pqr})$. At least one symbol must occur in three brackets. If $i$ occurs in $g_1$, $g_2$, $g_3$ but not in $g_4$, it occurs in $Q$ and can be brought into the fourth bracket by $[g_4, i]$. Hence we may suppose that each $g_n$ contains $i$. The remaining pairs of symbols in the brackets consist of four distinct pairs chosen from among $j$, $k$, $l$, $m$. If $j$ occurs in three brackets the form is of type $(ijk)(ijl)(ijm)(ikl)Q$, and reduces by $[g_3, k \text{ of } g_4]$. If no symbol $j$, $k$, $l$, $m$ occurs in more than two brackets the form is an invariant of type $(ijk)(ijl)(ikm)(ilm)$, which is seen to be reducible by squaring the identity

$$(ijk)(ilm)-(ijl)(ikm) \equiv (ijm)(ilk).$$

Since these arguments are equally valid for the dual forms, the system determined above is complete. To make it more accessible the forms are summarized in the following table, which gives, for forms of assigned orders and degrees, the number of forms in the irreducible sets, with a reference to the sets from which they arise.

*The complete system of five ternary quadratics*

(*a*) Forms of orders (2, 0), (0, 1), (1, 2):

| total degree | partial degrees | orders (2, 0) | orders (0, 1) | orders (1, 2) |
|---|---|---|---|---|
| 1 | {1} | 1 [1] | — | — |
| 4 | {31} | — | — | 1 [34] |
| | {2²} | 1 [15] | — | 1 [39] |
| | {21²} | 1 [6] | 1 [18] | 3 [36], [53] |
| | {1⁴} | 2 [70] | 3 [66] | 7 [57] |
| 7 | {3²1} | — | 1 [20] | — |
| | {32²} | — | 1 [48] | — |
| | {321²} | 1 [11] | 2 [22] | — |
| | {31⁴} | 4 [73] | 3 [76] | — |
| | {2³1} | 2 [41] | 3 [50] | — |
| | {2²1³} | 7 [12], [72] | 7 [24], [71] | — |

(*b*) Forms of orders (1, 0), (0, 2), (2, 1), (1, 3):

| total degree | partial degrees | orders (1, 0) | orders (0, 2) | orders (2, 1) | orders (1, 3) |
|---|---|---|---|---|---|
| 2 | {2} | — | 1 [2] | — | — |
| | {1²} | — | 1 [14] | 1 [38] | — |
| 5 | {32} | — | — | 1 [35] | — |
| | {31²} | 1 [47] | — | 1 [26] | — |
| | {2²1} | 1 [19] | 1 [7] | 2 [37] | — |
| | {21³} | 4 [49], [79] | 2 [40] | 3 [27] | 1 [42] |
| | {1⁵} | 10 [80] | 6 [75] | 5 [77] | 4 [56] |
| 8 | {3²2} | 1 [21] | — | — | — |
| | {3²1²} | 1 [60] | — | — | — |
| | {32²1} | 2 [23] | — | — | — |
| | {321³} | 3 [61] | — | — | — |
| | {2⁴} | 3 [67] | — | — | — |
| | {2³1²} | 7 [25], [55] | — | — | — |

(*c*) Forms of orders (0, 0), (3, 0), (0, 3), (1, 1):

| total degree | partial degrees | orders (0, 0) | orders (3, 0) | orders (0, 3) | orders (1, 1) |
|---|---|---|---|---|---|
| 3 | {3} | 1 [4] | — | — | — |
| | {21} | 1 [5] | — | — | 1 [3] |
| | {1³} | 1 [64] | 1 [51] | 1 [16] | 2 [45] |
| 6 | {3²} | — | 1 [29] | 1 [28] | — |
| | {321} | — | 1 [31] | 1 [30] | 1 [8] |
| | {31³} | — | 1 [62] | 1 [43] | 3 [58] |
| | {2³} | 1 [65] | 1 [17] | 1 [52] | 2 [46] |
| | {2²1²} | 1 [10] | 2 [33] | 2 [32] | 6 [9], [54] |
| | {21⁴} | 2 [68] | 3 [63] | 3 [44] | 16 [59], [78] |
| 9 | {3²1³} | 1 [74] | — | — | — |
| | {32²1²} | 1 [13] | — | — | — |
| | {2⁴1} | 2 [69] | — | — | — |

## THE IRREDUCIBILITY OF THE SYSTEM

72. The system of forms just tabulated is certainly complete, but the work which has preceded gives no evidence, in itself, that it does not include reducible forms. In order to demonstrate the irreducibility different methods have to be employed. The method adopted here is the same as that used to demonstrate the irreducibility of the complete system of four quadratics (Todd 1948). Since the discussion of the system of four quadratics contained incidentally, almost all the necessary results needed to establish the irreducibility of the system of five quadratics, the ideas will be reviewed here quite briefly; for the details the paper just cited may be referred to.

The method depends on the *S*-function analysis developed by Littlewood (1944, 1947), and a knowledge of his notation will be assumed. Any concomitant of a number of ternary quadratics, of degree $r$, determines a partition $[\mu]$ of $2r$ into three (or fewer) parts $p_1, p_2, p_3$; these integers being determined, when the order $n_1$ and the class $n_2$ of the form are known, by the relations
$$p_1+p_2+p_3 = 2r, \quad p_1-p_2 = n_1, \quad p_2-p_3 = n_2.$$
The form is then said to be of type $\{\mu\}$. Littlewood shows that the linearly independent concomitants of the quadratics of degree $r$ fall into a number of primitive sets. Each such set corresponds to a partition $[\nu]$ of $r$, and the forms belonging to such a set are said to be of class $\{\nu\}$. The number of primitive sets of class $\{\nu\}$ consisting of forms of type $\{\mu\}$ is the coefficient of the *S*-function $\{\mu\}$ in the expression $\{2\}\otimes\{\nu\}$.

It follows that the classes to which the linearly independent forms of type $\{\mu\}$ belong can be determined, for given $\{\mu\}$, by picking out from among the partitions of $r$ these partitions $\{\nu\}$ for which $\{\mu\}$ occurs in $\{2\}\otimes\{\nu\}$. For small degrees these can be read off from a table, while for larger degrees they may be found by calculation, as explained in detail in Littlewood (1947). These classes comprise reducible as well as irreducible forms. Now if two forms $C_1, C_2$ belong to classes $\{\nu_1\}, \{\nu_2\}$ the product $C_1 C_2$ is expressible linearly in terms of forms of classes $\{\nu\}$ which occur in the product $\{\nu_1\}\{\nu_2\}$. Hence, if forms are considered in increasing order of degree we can determine, among the classes $\{\nu\}$ which correspond to a concomitant of type $\{\mu\}$, all those which can conceivably represent reducible forms. The remaining classes, if such exist, will certainly represent irreducible forms. The system determined in this way, for the various types $\{\mu\}$, is not *a priori* necessarily complete, since owing to the possible existence of syzygies some of the classes which apparently correspond to reducible forms may prove to be irreducible. But it is certain that this system, so far as it goes, is irreducible.

It is evidently only necessary to consider types which appear in the table of the complete system given in § 71. For all these types the set of classes which certainly represent irreducible forms is known. In fact, all the forms of degree eight or less are of a type which occurs for four quadratics, and the corresponding classes were obtained (for any number of quadratics) in Todd (1948). And the invariants of degree nine—the only remaining type—were discussed by Littlewood (1947), who proved the existence of the single irreducible class $\{3^2 1^3\}$. Combining the results of these two papers, the existence of irreducible concomitants of the following classes and types is known.

| degree | orders | classes | degree | orders | classes |
|---|---|---|---|---|---|
| 1 | (2, 0) | $\{1\}$ | 5 | (1, 0) | $\{31^2\}+\{21^3\}$ |
| | | | | (0, 2) | $\{2^2 1\}+\{1^5\}$ |
| 2 | (0, 2) | $\{2\}$ | | (2, 1) | $\{32\}$ |
| | (2, 1) | $\{1^2\}$ | | (1, 3) | $\{21^3\}$ |
| | | | 6 | (0, 0) | $\{2^3\}+\{1^6\}$ |
| 3 | (0, 0) | $\{3\}$ | | (3, 0) | $\{3^2\}$ |
| | (3, 0) | $\{1^3\}$ | | (0, 3) | $\{3^2\}$ |
| | (0, 3) | $\{1^3\}$ | | (1, 1) | $\{321\}+\{31^3\}+\{2^2 1^2\}+\{21^4\}$ |
| | (1, 1) | $\{21\}$ | 7 | (2, 0) | $\{321^2\}+\{31^4\}$ |
| | | | | (0, 1) | $\{3^2 1\}+\{2^2 1^3\}$ |
| 4 | (2, 0) | $\{2^2\}$ | 8 | (1, 0) | $\{3^2 2\}+\{2^3 1^2\}$ |
| | (0, 1) | $\{21^2\}$ | | | |
| | (1, 2) | $\{31\}+\{21^2\}+\{1^4\}$ | 9 | (0, 0) | $\{3^2 1^3\}$ |

All these classes arise for five quadratics or less with the exception of the class $\{1^6\}$ of invariants of degree 6 (which gives the alternating invariant of six quadratics).

Now if $\{\nu\}$ is a class corresponding to a primitive set of concomitants of degree $r$, and if $[\rho]$ is a partition of $r$, the number of forms of the class $\{\nu\}$ whose partial degrees are given by the partition $[\rho]$ is the coefficient of $\{\nu\}$ in the product $\{\rho_1\}\{\rho_2\}\ldots\{\rho_r\}$ of the $S$-functions corresponding to the members of the partition $[\rho] = [\rho_1, \rho_2, \ldots, \rho_r]$. These coefficients are easily calculated and show that the number of forms of assigned type and partial degrees given by the classes in the table above coincides precisely with the number of forms of the same type and partial degrees in the table of the complete system. Thus the table of irreducible classes is complete, and the complete system of forms is irreducible.

Since, by Peano's theorem, all concomitants of any number of ternary quadratics belong to types arising for five or fewer quadratics together with the alternating invariant of six quadratics, the table of the present section gives, essentially, the complete system of any number of ternary quadratic forms, though it does not list the members of the system explicitly. The numbers of forms of given type and partial degrees can be calculated in the way just mentioned, but it does not seem worth while to give any numerical results, as all the systems for five or more quadratics contain a large number of forms (even though these belong to a relatively small number of classes).

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